A GUIDE TO Practical Seismology

Bogdan Felix Apostol and Liviu Cristian Cune

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1 Introduction

1.1 The Earth

The Earth is approximately spherical, with a mean radius $R = 6370km$, a very small flattening $(+7/-15km)$, mass $\simeq 6 \times 10^{24}kg$, and an average density $5.5g/cm^3$; the law of gravitational attraction is $\mathbf{F} = GmM\mathbf{r}/r^3$, where F is the force directed along the separation distance $\mathbf{r}$ between two point bodies with mass m and M; and $G = 6.67 \times 10^{-8}cm^3/g \cdot s^2$ is the gravitation constant.

Little is known about Earth's interior. The drilling down into the earth reaches at most $10-15km$. It is accepted that the Earth consists of several shells. First, at the surface, there is a solid crust, extending down to approximately $70km$, on average; locally it may have $5km$ thickness. Down to approximately $3000km$ an extremely viscous mantle exists. The next $2000km$ down to the centre are occupied by a liquid outer core. Finally, a solid inner core exists at the centre. The viscosity of the mantle is $10^{22} - 10^{25}g/cm \cdot s$; for comparison, water has $10^{-2}g/cm \cdot s$. The physical properties are discontinuous at the boundaries of these layers, like the "Moho" discontinuity (named after Mohorovicic) between crust and mantle, or the Gutenberg discontinuity between mantle and the liquid core, or the Bullen (or Lehmann) discontinuity between the inner core and the outer core. Changes in propagation of the elastic (seismic) waves have been measured and have indicated such discontinuities. The main chemical elements in the Earth's shells are Fe, O, Si, Mg, S, Ni, Co, Al. The crust is made mainly of silicon dioxide and aluminium oxide. The crust density is $3g/cm^3$; the inner core has probably the density $13g/cm^3$. The crust and the upper part of the mantle are called lithosphere; tectonic plates are located there, in slow motion. This motion is known as the continental drift. The largest rate of the continental drift seems to be $2.5cm/year$ (separation of the Americas from Europe and Africa).[1] The Earth's crustal movements are measured today by satellites in the Global Positioning System (GPS). Very likely, the earthquakes, volcanoes and mountains are produced at plate boundaries. The gravitational acceleration at the Earth's surface $g = 9.8m/s^2$ is preserved down to $3000km$, where it decreases appreciably. Earth's magnetic field is produced probably by convection and motion of electrical charges in the liquid outer core; the Earth's temperature is due probably to radioactive decays. In the inner core the temperature is probably $6000K$ and the pressure is $3.5 \times 10^{12} dyn/cm^2$.

A great deal of physical phenomena can be observed and even quantitatively measured, related to the internal motion in the Earth. Among these are heat flow, quasi-static displacement (the motion of the tectonic plates), strain, variations in gravity, electromagnetic phenomena; and, of course, seismic waves. The range of these variables is huge. Explosive charges are detected from $1g$ to $10^9 kg$; ground displacements are measured from 1μ to tens of meters for the slip of a major fault during an earthquake. Earthquakes vary in intensity over more than 18 orders in energy (one of the greatest earthquake was the Chilean earthquake of 1960, May 22). Seismic networks vary from tens of meters for an engineering foundation survey to $10^4 km$ for the global array of seismological observatories.

1.2 Seismic sources and waves

The Earth is a deformable body, which may bear local elastic movements, like static deformations and elastic waves. Among these, the greatest are the earthquakes. The Earth may be viewed as an elastic body, in the first approximation an infinite homogeneous and isotropic elastic medium; however, the effects of the earthquakes are felt on its surface, so the surface should be included.

The mechanics of deformable bodies and the theory of elasticity appeared gradually, over more than two hundred years. About 1660

[1] A. Wegener, "Die Herausbildung der Grossformen der Erdrinde (Kontinente und Ozeane) auf geophysikalischer Grundlage", Petermanns Geographische Mitteilungen **63** 185, 253, 305 (1912); *Die Entstehung der Kontinente und Ozeane*, Vieweg & Sohn, Braunschweig (1929).

Hooke determined the proportionality of the force with the deformation: *ut tensio, sic vis*. In 1821 Navier established the equilibrium of the elastic bodies and their vibrations; Young and Fresnel showed the polarization of the waves, in relation to the transverse polarization of light; in 1822 Cauchy introduced the strain and the stress tensors; then, Poisson determined the compression and the shear elastic waves, and Green introduced the strain-energy function; Kelvin computed the static deformation produced by a localized force and Stokes derived the elastic waves from a localized force in an infinite elastic medium. Elastic waves and vibrations were studied intensively by Rayleigh, Lamb and Love at the end of the 19th century and the beginning of the 20th century. Non-linear elasticity (also called "finite" elasticity), or the mechanics of deformable visco-elastic media, or of micro-structured media are various generalizations.

It is widely accepted that the earthquakes are produced by a sudden release of the elastic energy built up locally at the boundary of two or more tectonic plates; such an interface of tectonic plates, where a rupture in the material may occur, is a fault. The spontaneous slip occurring in a fault, or explosions, is a seismic source. The force acting in a faulting source is usually related to the fault slip over a finite faulting area. For a "volume" source, like those associated with explosions, the force is related to the dilatational strain. The focus of an earthquake is localized in a small volume, of various shapes which, however, are irrelevant as long as the focal volume may be viewed as being concentrated in a point. In the first approximation, the fault slip, occurring in the focus, is characterized by a direction; the pressure, caused by an explosion for instance, is the (uniform) force per unit area on the surface enclosing a localized, small volume. The focus is active a certain duration of time (which often may be taken as an impulse-like duration). In general, a faulting slip generates tensorial forces; a faulting source is characterized by a tensor; this is the tensor of the seismic moment. The "volume" sources correspond to an isotropic tensor of seismic moment (isotropic sources). As long as the material remains elastic the seismic forces generate a slow movement of the tectonic plates; such a quasi-static deformation relieves the stress and diminishes the probability of an earthquake. If the material yields, and a sudden rupture appears, the seismic forces generate a shock and cause an earthquake. Hence, monitoring the small displacement of the tectonic plates, especially in regions prone to earthquakes, may give an indication about the likelihood of an earthquake.

The rock fracture and the faulting mechanism for seismic sources (located from a few kilometers beneath the Earth's surface down to $700km$) have been accepted gradually, especially after the big earthquakes of Mino-Owari (Japan), 1891 and San Francisco, 1906. In a big earthquake soil displacement as large as a few meters may appear, both horizontally and vertically, along distances as large as tens to hundreds of kilometers,[2] and accelerations may exceed the gravitational acceleration; sometimes, the soil displacement is permanent. An extensive phenomenology of the earthquakes was given by Richter.[3]

Although Earth is not a perfect elastic body, it is still approximated by a simple, homogeneous and isotropic elastic solid, with two elastic moduli: the Lame coefficients λ and μ, or the Young modulus E and the Poisson ratio σ.[4] In such an (infinite) elastic solid two kinds of elastic waves can be propagated: longitudinal waves, associated with the compressibility of the solid, and transverse waves, associated with the shear elastic properties of the solid. The longitudinal waves propagate faster; they are called P waves ("primary", compressional waves); their velocity (in the crust) is approximately $7km/s$; the transverse waves are slower, they are called S waves ("secondary", shear waves); their velocity in the crust is approximately $3km/s$.[5] In general, the velocity of the elastic wave seems to increase with the increasing depth in the earth.

The problem of propagation of the seismic waves is complicated by the presence of the Earth's surface, which suggests to approximate the Earth by an elastic half-space (we may neglect in the first approximation the Earth's surface curvature). The equations of motion of the linear elasticity are second-order differential equations in

[2] H. F. Reid, *Mechanics of the Earthquake, The California Earthquake of April 18, 1906*, vol. 2, Carnegie Institution, Washington (1910).

[3] C. F. Richter, *Elementary Seismology*, Freeman, San Francisco, CA (1958).

[4] We may take as mean values for Earth $E \simeq 10^{11} dyn/cm^2$, $\mu = E/2(1+\sigma) \simeq 10^{11} dyn/cm^2$, $\lambda = E\sigma/(1-2\sigma)(1+\sigma) \simeq 10^{11} dyn/cm^2$ (Poisson ratio $\sigma \simeq 0.2$).

[5] The notations P and S seem to originate historically in "primary" and "secondary" waves. Both will be called here primary waves.

variables time and position (elastic waves equation, known also as the Navier-Cauchy equation). Such an equation is amenable to two distinct approaches. First, we may consider a source appearing at a certain moment of time, which produces waves. Before, there is no motion. This is the propagating-wave approach, governed by the causality principle: the waves are produced only by sources acting in the past. The waves propagate in the future. The wave source lasts a finite time, usually a short time; in this case it produces localized waves, like the P and S seismic waves, which have a spherical-shell structure. This is the case of earthquakes. Once arrived at Earth's surface, these seismic waves (primary seismic waves) produce surface sources of secondary waves, according to Huygens principle, propagating on the surface, which gives the seismic main shock (actually, two main shocks, corresponding to the two distinct velocities of the seismic waves). The seismic main shock is a delocalized wave propagating back in the Earth. On the surface it looks like a propagating wall, behind the primary waves, with a long tail (actually two walls, corresponding to the two P and S waves, for distinct components of the elastic displacement). Despite having a long tail, the main shock is a wave, propagating from a certain instant towards the future. This is a transient regime, seen and felt in earthquakes. The inner layers of the Earth may cause dispersion of the elastic waves.[6] After a while, the seismic waves suffer multiple reflections on the spherical Earth's surface, their presence is continuous on the whole Earth's surface, and they produce free oscillations of the Earth (eigenoscillations, eigenvibrations, normal modes); the seismic source ceased its action since long. The frequencies of the P and S seismic waves are, mainly, of the order $1s^{-1}$ (it seems that the periods of the seismic waves are in the range $0.1s$-$10s$, though these are only orientative figures); their wavelength is longer than the dimensions of the focal region. The free oscillations of the Earth may have much lower frequencies.

The second approach consists in viewing a continuous source, whose effects (waves) act continuously upon the whole Earth's surface. They should obey boundary conditions on the surface. The time is indefinite in this approach, the role of the causality principle is played

[6] A. E. H. Love, *Some Problems of Geodynamics*, Cambridge University Press, London (1911).

now by the boundary conditions. This is the vibration approach. In vibration, the waves propagate both in the future and in the past, their superposition gives a vibration. A source may produce forced vibrations, in the absence of a source the vibrations are free (free oscillations, eigenoscillations). The vibration regime is a stationary regime, distinct from the propagating-wave transient regime. In 1885 Rayleigh discovered that an important contribution to the vibrations of a homogeneous and isotropic elastic half-space with a free plane surface is brought by damped waves, called Rayleigh's surface waves (actually vibrations).[7] Since they last long, over large distances (as any vibration), it was tempting to associate them with the seismic main shock. The Rayleigh's surface waves propagate as plane waves along the surface and are damped vibrations along the direction perpendicular to the surface; in another nomenclature they may be called guided waves.

The recording of the seismic waves in seismograms shows approximately, very approximately, a general, qualitative picture of P and S elastic waves and main shock (shocks). The first attempt at constructing a theoretical seismogram was done by Lamb in 1904 for a seismic source on the surface of a homogeneous and isotropic elastic half-space or buried in such a half-space;[8] Lamb's results consist of a sequence of three pulses, which Lamb associated to a preliminary feeble tremor (say, P, S waves) and Rayleigh surface "waves", according to their arrival times (the surface waves are the slowest). In fact, Lamb's solution is for a vibration problem, not for a propagating wave problem. For a temporal-impulse source, Rayleigh's and Lamb's solution (though inapplicable) extends over the whole free surface, much before the arrival of a main shock, albeit exhibiting a propagating double-wall structure.[9] Even the seismic main shock, though extended over the surface, is not a vibration, because its source, the source of secondary waves, lies just on the surface, and boundary conditions are meaningless in this case. On the other hand, a construction

[7] Lord Rayleigh, "On waves propagated along the plane surface of an elastic solid", Proc. London Math. Soc. **17** 4 (1885).

[8] H. Lamb, "On the propagation of tremors over the surface of an elastic solid", Phil. Trans. Roy. Soc. (London) **A203** 1 (1904).

[9] B. F. Apostol, "On the Lamb problem: forced vibrations in a homogeneous and isotropic elastic half-space", Arch. Appl. Mech. **90** 2335 (2020).

built on Earth's surface, under the action of the seismic waves may suffer vibrations, of course. The first seismogram was recorded in the early 1880s. P, S and a main shock (interpreted as surface waves) have been first recognized on a seismogram by Oldham in 1900.[10] Reflections of the original seismic pulses in Earth's surface layers may generate long-lasting oscillations; for a long time Earth vibrates and oscillates (rather than "radiates"), as Jeffreys discussed in 1931;[11] seismic waves may be localized in Earth's surface layers. In general, the scattering of the seismic waves by inhomogeneities makes them to last long. The seismograms exhibit a characteristic, long tail (coda).

One of the main problems of the Seismology is to understand the seismograms, *i.e.* the seismic movement recorded at Earth's surface. This is known as the seismological problem. Typically, any seismogram exhibits a preliminary tremor of feeble movement, which consists of spherical-shell P- and S-seismic waves, followed by a main shock (or two main shocks); the main shock finally subsides slowly with a long seismic tail. The recorded pattern exhibits many oscillations. Spherical waves are generated by seismic sources localized in space and time. These waves interact with the Earth's surface and produce additional sources moving on the Earth's surface. These surface sources generate secondary waves, which propagate in the whole Earth; on the surface they cause the main shock and its long tail. The oscillations are probably due to the internal structure of the earthquake focus, which may include successive and adjacent point ruptures (the structure factor of the earthquake focal region); also, inhomogenities of the Earth, multiple reflections and the seismographs' characteristics may play a role in these oscillations.

1.3 Empirical laws

The energy released by an earthquake was, and still is, estimated by the damage produced by the seismic waves at Earth's surface, in

[10] R. D. Oldham, *Report on the Great Earthquake of 12th June, 1897*, Geol. Surv. India Memoir 29 (1899); "On the propagation of earthquake motion to long distances", Trans. Phil. Roy. Soc. London **A194** 135 (1900).

[11] H. Jeffreys, "On the cause of oscillatory movement in seismograms", Monthly Notices of the Royal Astron. Soc., Geophys. Suppl. **2** 407 (1931).

the region affected by the earthquake. The highest known value is of the order $10^{30}erg$, for the 1960 Chilean earthquake, or the 1964 Alaskan earthquake. The smallest value is about $10^{12}erg$ for micro-earthquakes; $10^{5}erg$ corresponds to micro-fractures in laboratory experiments on loaded rock samples. For such large variations a logarithmic scale is convenient. The earthquake magnitude M appeared this way, defined by

$$E/E_0 = e^{bM} \quad , \tag{1.1}$$

where E_0 is a threshold energy and b is a constant, chosen by convention $b = 3.45$.[12] Originally, the law was written with powers of ten, where $b = 3/2$. By another convention, the parameter E_0, measured in erg, is given by $\lg E_0 = 15.65$ (decimal logarithm). The logarithmic form of this law is also known as the Gutenberg-Richter law.[13] Later on, the earthquake energy was associated with the tensor of the seismic moment, and a similar logarithmic law was introduced for the magnitude of this tensor, which defines a so-called earthquake moment magnitude. This law is called the Hanks-Kanamori law.[14]

It was shown empirically that the number of earthquakes with magnitude greater than M which appear in a given region in a given time duration obeys a logarithmic law

$$\ln N = const - \beta M \quad , \tag{1.2}$$

where β is a constant which depends on the data set (like *const* too). A reference value $\beta = 2.3$ is accepted (in decimal logarithms $\beta = 1$). This is a statistical law. It is called also the Gutenberg-Richter law (known also as the excedence, or cumulative law).

It was noticed that a big earthquake is often preceded and succeeded by many smaller earthquakes, which appear in the same seismic region

[12] T. Utsu and A. Seiki, "A relation between the area of aftershock region and the energy of the mainshock" (in Japanese), J. Seism. Soc. Japan **7** 233 (1955); T. Utsu, "Aftershocks and earthquake statistics (I): some parameters which characterize an aftershock sequence and their interaction", J. Faculty of Sciences, Hokkaido Univ., Ser. VII (Geophysics) **3** 129 (1969).

[13] B. Gutenberg and C. Richter, "Frequency of earthquakes in California", Bull. Seism. Soc. Am. **34** 185 (1944); "Magnitude and energy of earthquakes", Annali di Geofisica **9** 1 (1956) (Ann. Geophys. **53** 7 (2010)).

[14] H. Kanamori, "The energy release in earthquakes", J. Geophys. Res. **82** 2981 (1977); T. C. Hanks and H. Kanamori, "A moment magnitude scale", J. Geophys. Res. **84** 2348 (1979).

and in a reasonably short time interval; they are called foreshocks and aftershocks, respectively. The time distribution of these accompanying earthquakes is given approximately by

$$\frac{\Delta N}{\Delta t} \sim \frac{1}{const + t} \ , \tag{1.3}$$

where ΔN is the number of earthquakes which appear in the time interval Δt measured with respect to the main shock. This is known as Omori's law.[15]

Finally, it seems that the greatest aftershock of a main shock has a magnitude smaller by $\Delta M = 1.2$ than the main shock. This is known as Bath's law.[16]

All that we know about earthquakes reduces practically to these empirical, disparate laws. If we add the lack of knowledge of the P and S seismic waves and the main shock, we can see that we do not know much about earthquakes. It is claimed that the tensor of the seismic moment is related to the main shock, the later associated with surface waves; and, from measurements of these waves we may have information about the seismic moment and the magnitude of the earthquakes (through the Hanks-Kanamori law). Also, it is claimed that this knowledge is incorporated in numerical codes, released by various agencies, to be used for the determination of the seismic moment, the magnitude and, possibly, other earthquakes parameters. However, this knowledge is not in the public domain and, consequently, cannot be checked.

[15] F. Omori, "On the after-shocks of earthquakes", J. Coll. Sci. Imper. Univ. Tokyo **7** 111 (1894).

[16] M. Bath, "Lateral inhomogeneities of the upper mantle", Tectonophysics **2** 483 (1965); C. F. Richter, *Elementary Seismology*, Freeman, San Francisco, CA (1958) p. 69.

1.4 Seismology

Recently, basic results have been obtained in Seismology,[17] which are described briefly below.

The law of energy accumulation with increasing time in a pointlike focus was obtained by using the continuity equation and the energy conservation. The geometric-growth parameter r has been introduced ($1/3 < r < 1$), which makes the difference between $\beta = br$ and b. Also, besides the energy threshold E_0, the basic time threshold t_0 has been introduced, which gives the seismicity rate $1/t_0$.

Based on this law, the time and energy probability distributions have been derived, as well as the standard Gutenberg-Richter magnitude distributions. This way, the background seismicity of regular earthquakes has been defined and Omori's law of conditional probability has been derived. These laws have been applied to the the estimation of the mean recurrence time of earthquakes and to the analysis of next-earthquakes distribution, both results of practical relevance.

The law of energy accumulation and the time and magnitude distributions have been used to derive the bivariate distributions, which account for the earthquake correlations in the foreshock-main shock-aftershock sequences. Time-magnitude, dynamical and statistical correlations have been highlighted, and correlation-modified Gutenberg-Richter distributions have been derived. This way, Bath's law has been explained and a procedure was established for estimating the occurrence time of a main shock, by using the analysis of the foreshocks. In the vicinity of a main shock the parameter β decreases, while the number of aftershocks and the parameter β increase after a main shock, as a result of the change in the seismicity conditions brought about by the main shock.

The tensorial force acting in a pointlike focus has been established and the notion of the elementary earthquake (temporal-impulse force) has been introduced. This force is governed by the tensor M_{ij} of the seismic moment. The elastic wave equation has been solved with this

[17] B. F. Apostol, *The Theory of Earthquakes*, Cambridge International Science Publishers, Cambridge (2017); *Introduction to the Theory of Earthquakes*, Cambridge International Science Publishers, Cambridge (2017); *Seismology*, Nova, NY (2020.

force, as well as the elastic equilibrium equation (Navier-Cauchy equations). This way, the P and S seismic waves have been derived, the transient regime of the seismic waves has been established, and the static deformations at Earth's surface have been computed. A regularization method has been introduced for the solutions of the elastic wave equation and a new method has been introduced for computing static deformations of an elastic half-space with general forces.

The main shock produced by the seismic waves, especially on Earth's surface, has been computed, and its singular wavefront has been derived. Thus, together with the derivation of the P and S seismic waves, the structure of the seismograms has been explained. This is known as the seismological (or Lamb's) problem. The identification of the transient regime was instrumental in solving this problem.

The above results have been applied to the seismological inverse problem, *i.e.* the derivation of the tensor of the seismic moment. The seismic moment was derived from the P and S seismic waves measured at Earth's surface, by using the Kostrov representation and the energy conservation. On this occasion, all the parameters of the seismic source were derived from empirical measurements of the P and S waves, like the duration of the seismic activity in the earthquake focus, the dimension of the focus, the focus strain and its rate, the orientation of the fault and the slip along the fault. The relation $\left(M_{ij}^2\right)^{1/2} = 2\sqrt{2}E$ between the magnitude of the seismic moment and the energy (E) has been established. Also, the results have been applied to explosions, which have an isotropic tensor of the seismic moment. It was shown that a hybrid focal mechanism, which would imply a shear faulting and an isotropic dilatation (or compression) is impossible. A procedure has also been devised for getting the seismic moment from the (very small) static displacements measured (theoretically) in the epicentral region.

Finally, the effect of the seismic movements on the constructed structures on Earth's surface has been investigated by using the models of the embedded bar, the buried bar and coupled oscillators (and bars). Amplification factors have been derived and the risk brought by soft inclusions, and, in general, by inhomogeneities in elastic structures was highlighted. A new method for computing the elastic vibrations of a half-space has been introduced and the difference between propagating waves and vibrations was emphasized; also, two-dimensional related problems were solved.

Most of these subjects can also be found in a previously published book.[18] Apart from new, original points, the present book emphasizes the practical side of such problems of seismological interest.

1.5 Description of the book

The first part of the book is devoted to statistical Seismology (the first four chapters). The geometric-growth model for energy accumulation in the focus is introduced, the time distribution of earthquakes is established and the standard Gutenberg-Richter earthquake distribution in magnitude is deduced. Conditional probabilities are described and the Omori distribution is derived. Aplications to Vrancea earthquakes are presented. It is shown that the background seismicity (consisting of regular earthquakes) of a region over a long period of time is characterized by two parameters: the slope of the Gutenberg-Richter distribution and the seismicity rate. The recurrence time (periodicity problem) is governed by these parameters. Further on, the next-earthquake distributions are introduced, and their practical relevance is emphasized. A main subject in statistical Seismology is the earthquake correlations, which characterize the distribution of the foreshocks and the aftershocks (accompanying seismic events). The pair correlations are established, the roll-off effect is explained and the nature and characteristics of the accompanying seismic events are presented. The pair correlations explain the deviations from the standard Gutenberg-Richter distribution. On this occasion the Bath's law is explained. Short sequences of correlated foreshocks may be used to make a short-term prediction of a main shock, including its occurrence moment and magnitude. A few examples of application of this procedure are given. Also, in this part the problem of the statistical equilibrium of a seismic zone is discussed, by means of the earthquake entropy. It is shown that the seismic activity is a non-equilibrium process, where the steadily decreasing entropy is interrupted from time to time by abrupt increases, due to big earthquakes.

[18] B. F. Apostol, *Seismology*, Nova, NY (2020).

The second part of the book deals with seismic waves, seismic moment and the static deformations (three chapters). First, the tensorial seismic point force is introduced. This is a temporal-impulse force, localized in space and governed by the tensor of the seismic moment. It is shown how the P and S seismic waves can be derived from this force, and how these primary waves generate on Earth's surface sources of secondary waves, which generate the main shock(s). The primary seismic waves are scissor-like spherical shells, and the main shock comes out with an abrupt wall and a long tail, propagating on the surface behind the primary waves. This way, the seismological problem is solved. A complete solution would require the determination of the seismic-moment tensor and the other parameters of the earthquakes by measurements performed on Earth's surface. We show that the amplitude of the P and S seismic waves provides a means of determining this tensor. The determination procedure is conducted in a consistently covariant way. It gives access to the earthquake energy and magnitude, as well as to the dimension of the focus, the duration of the seismic activity in the focal region and the orientation of the fault. The introduction of the seismic tensorial force raises another problem, besides the seismological problem: the computation of the static deformations generated on Earth's surface. The result of these computations is presented and discussed in this book. Moreover, the periodic small discharge of the seismic stress generates small deformations in the epicentral zone. We show, on one side, how to estimate an average seismic moment and the depth of the focus by measuring such quasi-static deformations; and, on the other side, we show that a continuous monitoring of the quasi-static epicentral deformations may give information about a possible evolution of the seismic activity, because, after periods of silence we may expect a burst of seismic activity.

The next three chapters of the book deal with local seismic effects. An important problem in Seismology is how to secure the buildings erected on Earth's surface against the destructive action of the earthquakes. We show that the buildings may be viewed as vibrating bars, which, under the seismic action may resonate; also, sub-surface inhomogeneities may behave as resonating embedded (buried) bars. In both cases we get local amplification factors, evaluated in the book.

The estimation of the local effects requires the understanding of the relevance of the site spectral response, which is currently measured on Earth's surface. We discuss the information the site response may give, in the complex context of the presence of the inhomogeneities, different local velocities of the elastic waves, or different wave polarizations. A special chapter is devoted to the interaction of a harmonic-oscillator model with an elastic wave, the associated amplification factors, the role of damping, the different behaviour under seismic shocks and periodic elastic waves.

Hitherto, besides giving theoretical tools, sometimes in great detail, necessary for understanding the seismic phenomenon, we focused on practical procedures relevant for Seismology. The book describes in detail activities (tasks) which can be undertaken by anybody in order to get knowledge about earthquakes. For instance, by updating periodically the background earthquake activity, we may get information about the recurrence time of the big earthquakes; the next-earthquake distribution may answer our current question: "what may we expect after an important earthquake?"; by fitting (daily, sometimes hourly) the correlated foreshock sequences we may predict, on a very short term, a main seismic shock; by updating periodically the earthquake entropy we may see if something changed in the seismic non-equilibrium process, so we may get an insight on what we may expect in the future in regard to earthquakes. By using the procedure described in this book we may determine the seismic moment, the magnitude and the characteristics of the fault; or, we may assess the behaviour of the small epicentral seismic discharges. By measuring the local effects, the amplification factors and the site spectral response, we may get information about improving the design of the constructions. By using such practical procedures, the Seismology reveals its practical side, besides its pure-science side, which may become useful in our daily life.

The end part of the book deals with a few more theoretical issues (three chapters). Such issues are not of practical use, but they may help us to better understanding of the problems raised by the seismological studies. The first problem tackled in this context is the problem of the inhomogeneities dispersed naturally in the Earth. We give a detailed presentation of this problem, regarding both bulk and surface inhomogeneities (rough surface). We show that the bulk inhomogeneities may renormalize the velocity of the elastic waves, may produce a slight distortion of the localized seismic waves and may produce dispersion of the waves, as expected. In general, such effects have, relatively, little relevance, though, in some cases, a rough surface may generate the surprising effect of a surface localized wave. The next problem is the vibration problem. In the seismological studies the wave propagation and the elastic vibrations are treated to some extent indistinctly. We introduce a new mathematical technique of treating the elastic vibrations, which serves to distinguish them sharply from propagating waves, and apply this technique to a half-space. In this context we discuss amply the so-called Lamb problem. Further on, we dedicate a rather long space to the elastic vibrations of a sphere, which is an old and important subject in Seismology. We are concerned in particular with the effect of gravitation and rotation and with approximate techniques of estimating vibrations in the case of the Earth, which has a large radius. Finally, we devote the end chapter to a rather academic problem, concerning the seismic phenomenon in two dimensions. Though far from a direct physical relevance, this problem is often discussed because, in some cases, it may look more simple than the three dimensional problem. We compute the seismic waves in two dimensions, the main shock and the vibrations of an elastic half-plane.

We hope, according to the above description, that the book may be useful, not only for an understanding of the seismic phenomena, but, especially, to those who wish to be active in Seismology.

We are very thankful to our colleagues in the Department of Engineering Seismology in the Institute for Earth's Physics and the Department of Theoretical Physics in the Institute for Physics and Nuclear Engineering, both at Magurele, for enlightening discussions, very useful suggestions, pointing out errors and offering corrections (many accepted and included) and a thorough critical reading of the manuscript.

2 Background Earthquakes

2.1 Geometric-growth model

We consider a typical earthquake (which we call a tectonic earthquake), with a small focal region localized deeply in the solid crust of the Earth. Also, we assume that the movement of the tectonic plates (rocks) leads to energy accumulation in this pointlike focus. (For surface earthquakes the focal mechanism may exhibit additional features, for instance a propagating focus. A propagating focus can also be associated with a deep extended shear faulting).[1]

The energy accumulation in the focus is governed by the continuity equation (energy conservation)

$$\frac{\partial E}{\partial t} = -\boldsymbol{v} gradE \ , \tag{2.1}$$

where E is the energy, t denotes the time and $\boldsymbol{v}$ is an (undetermined) accumulation velocity. For a localized focus we may replace the derivatives in equation (2.1) by ratios of finite differences. For instance, we replace $\partial E/\partial x$ by $\Delta E/\Delta x$, for the coordinate x. Moreover, we assume that the energy is zero at the borders of the focus, such that $\Delta E = -E$, where E is the energy in the centre of the focus. Also, we assume a uniform variation of the coordinates of the borders, given by equations of the type $\Delta x = u_x t$, where $\boldsymbol{u}$ is a small (undetermined) displacement velocity of the medium in the focal region. The energy accumulated in the focus is gathered from the outer region of the focus, as expected. We note that the displacement of the rocks in the focal region affects larger zones with increasing time. With these

[1] B. F. Apostol, *Theory of Earthquakes* and *Introduction to the Theory of Earthquakes*, Cambridge International Science Publishing, Cambridge (2017).

assumptions equation (2.1) becomes

$$\frac{\partial E}{\partial t} = \left(\frac{v_x}{u_x} + \frac{v_y}{u_y} + \frac{v_z}{u_z} \right) \frac{E}{t} \ . \tag{2.2}$$

Let us assume an isotropic compression without energy loss; then, the two velocities are equal, $\boldsymbol{v} = \boldsymbol{u}$, and the bracket in equation (2.2) acquires the value 3. In the opposite limit, we assume a one-dimensional compression. In this case the bracket in equation (2.2) is equal to unity. An energy loss may exist in this case, as a consequence of a back-displacement, off the focus along the other two directions, such that the bracket in equation (2.2) may have a value slightly smaller than unity. A similar analysis holds for a two dimensional accumulation process, such that, in general, we may write equation (2.2) as

$$\frac{\partial E}{\partial t} = \frac{1}{r} \frac{E}{t} \ , \tag{2.3}$$

where the parameter r varies in the range (1/3, 1).

The integration of this equation needs a cutoff (threshold) energy and a cutoff (threshold) time. We may imagine that during a short time t_0 a small energy E_0 is accumulated. In the next short interval of time this energy may be lost, by a relaxation of the focal region. Consequently, such processes are always present in a focal region, although they do not lead to an energy accumulation in a focus. We call them fundamental processes (or fundamental earthquakes, or E_0-seismic events). It follows that we must include them in the accumulation process, such that we measure the energy from E_0 and the time from t_0. The integration of equation (2.3) leads to the law of energy accumulation in the focus[2]

$$t/t_0 = (E/E_0)^r \ . \tag{2.4}$$

The time t in this equation is the time needed for accumulating the energy E, which may be released in an earthquake (the accumulation time).

[2]B. F. Apostol, "A Model of Seismic Focus and Related Statistical Distributions of Earthquakes", Phys. Lett. **A357** 462 (2006); "A Model of Seismic Focus and Related Statistical Distributions of Earthquakes", Roum. Reps. Phys. **58** 583 (2006).

2.2 Gutenberg-Richter statistical distributions

Let us assume a time interval $(0, t)$ divided in cells of duration t_0. In each cell we may have an event with energy E_0 (a fundamental process). We view these events as independent events. We have a total number t/t_0 of such cells, such that the probability to have an E_0-event in the interval (t_0, t) is t_0/t. On the other hand, let us assume that $P(t')dt'$ is the probability to have an earthquake with energy $E > E_0$ in the time interval $(t', t' + dt')$. Also, we view these earthquakes as independent events. The probability to have an earthquake with energy $E > E_0$ in the time interval (t_0, t) is $\int_{t_0}^{t} dt' P(t')$. Consequently, we have the equality

$$\frac{t_0}{t} + \int_{t_0}^{t} dt' P(t') = 1 \ , \tag{2.5}$$

which gives, by differentiation, the probability density

$$P(t) = \frac{t_0}{t^2} \ . \tag{2.6}$$

This is a single-event probability distribution of independent events. It is worth noting that $(t_0/t^2)dt$ is the probability to have an earthquake in the interval $(t, t + dt)$, with no other conditions regarding the time before the time moment zero and after the time duration t.

Making use of accumulation equation (2.4), we get from equation (2.6) the energy distribution [3]

$$P(E)dE = \frac{r}{(E/E_0)^{1+r}} \frac{dE}{E_0} \ . \tag{2.7}$$

At this point we may use the exponential law[4] $E/E_0 = e^{bM}$, where

[3] A power law $E^{-\alpha}$ for the energy distribution of earthquakes was suggested as early as 1932 by K. Wadati, "On the frequency distribution of earthquakes", J. Meteorol. Soc. Japan **10** 559 (1932), with an estimated exponent $\alpha = 0.7 - 2.3$.

[4] H. Kanamori, "The energy release in earthquakes", J. Geophys. Res. **82** 2981 (1977); T. C. Hanks and H. Kanamori, "A moment magnitude scale", J. Geophys. Res. **84** 2348 (1979); see also B. Gutenberg and C. Richter, "Frequency of earthquakes in California", Bull. Seism. Soc. Am. **34** 185 (1944); "Magnitude and energy of earthquakes", Annali di Geofisica **9** 1 (1956) (Ann. Geophys. **53** 7 (2010)).

M is the earthquake magnitude and $b = \frac{3}{2} \cdot \ln 10 = 3.45$ ($\ln 10 = 2.3$), by convention. This equation is the definition of the earthquake magnitude. Historically, it was assumed that the energy E released by an earthquake would be distributed, quasi-uniformly, in a region with the volume $\sim R^3$ (a sphere with radius R). According to the exponential formula $E/E_0 = e^{bM}$ we have $R^3 \sim 10^{bM}$ and $R \sim 10^{\frac{1}{3}bM}$, in powers of ten. The area of the surface covering this volume is $S \sim R^2 \sim 10^{\frac{2}{3}bM}$. It was assumed that an estimate of this area could be obtained from the area affected on Earth's surface by an earthquake, including its subsequent companions (aftershocks). By convention, it has been taken $\lg S = M + const$, which leads to $b = \frac{3}{2}$ (in decimal logarithms).[5] It is worth noting that such assumptions imply that an earthquake and its aftershocks (at least those in its immediate temporal neighbourhood) may be viewed as a single seismic event. The estimation of an earthquake energy from the damaged area on the surface of the Earth is still used today.

Making use of $E/E_0 = e^{bM}$ in equation (2.7) we get the (normalized) magnitude distribution

$$P(M)dM = \beta e^{-\beta M} dM \ , \tag{2.8}$$

where $\beta = br$. In decimal logarithms, $P(M) = \frac{3}{2}r \cdot 10^{-\frac{3}{2}rM}$, where $0.5 < \frac{3}{2}r < 1.5$ (for $1/3 < r < 1$). Usually, the mean value $\frac{3}{2}r = 1$ ($\beta = 2.3$) is used as a reference value, corresponding to $r = 2/3$.[6] We note that the magnitude zero corresponds to energy E_0, such that an E_0-event means, in fact, the absence of any earthquake.

The magnitude distribution can be used to analyze the empirical dis-

[5] T. Utsu and A. Seiki, "A relation between the area of aftershock region and the energy of the mainshock" (in Japanese), J. Seism. Soc. Japan **7** 233 (1955); T. Utsu, "Aftershocks and earthquake statistics (I): some parameters which characterize an aftershock sequence and their interaction", J. Faculty of Sciences, Hokkaido Univ., Ser. VII (Geophysics) **3** 129 (1969).

[6] S. Stein and M. Wysession, *An Introduction to Seismology, Earthquakes, and Earth Structure*, Blackwell, NY (2003); A. Udias, *Principles of Seismology*, Cambridge University Press, NY (1999); T. Lay and T. C. Wallace, *Modern Global Seismology*, Academic, San Diego, CA (1995); C. Froelich and S. D. Davis, *Teleseismic b values; or much ado about* 1.0, J. Geophys. Res. **98** 631 (1993).

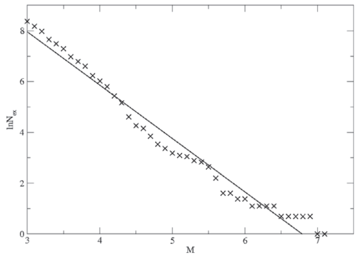


Figure 2.1: Excedence law $\ln N_{ex} = \ln N_0 - \beta M$ fitted to the cumulative distribution of Vrancea earthquakes, 1981 – 2018, $M \geq 3$, $\Delta M = 0.1$, for $\beta = 2.1$ and $-\ln t_0 = 10.62$.

tribution

$$P(M) = \frac{\Delta N}{N_0 \Delta M} = \frac{t_0 \Delta N}{T \Delta M} = \beta e^{-\beta M} \quad , \tag{2.9}$$

of ΔN earthquakes with magnitude in the range $(M,\, M + \Delta M)$ out of a total number $N_0 = T/t_0$ of events which occurred in time T. We note that we introduce here the seismicity rate $1/t_0$.

The parameter t_0 in equation (2.9) has the same meaning as the cutoff time t_0 used in the geometric-growth model (it is the same). Indeed, from the probability $\beta e^{-\beta M} dM$ we may adopt βM as a natural magnitude, *i.e.* we may measure the magnitudes in units $1/\beta$. With this convention we have a number $\Delta N/\beta\Delta M$ of earthquakes with the natural magnitude βM (*i.e.* the number of earthquakes per unit of natural magnitude), occurring in the mean recurrence time

$$\begin{aligned} t_r &= \tfrac{T}{\Delta N/\beta\Delta M} = \tfrac{T\beta\Delta M}{\Delta N} = \\ &= \tfrac{T\beta\Delta M}{N_0\beta\Delta M e^{-\beta M}} = \tfrac{T}{N_0} e^{\beta M} = t_0 e^{\beta M} \ . \end{aligned} \tag{2.10}$$

This is precisely the accumulation time for an earthquake with magnitude M, according to $t = t_0 e^{\beta M}$, where t_0 is the cutoff time in the accumulation law (equation (2.4)). Indeed, the mean recurrence time of the fundamental earthquakes with $M = 0$ ($E = E_0$) is given by this formula as $t_r(M = 0) = t_0$, which shows that the cutoff time ($t_r(M = 0)$, *i.e.* the accumulation time for the fundamental earthquakes with zero magnitude) is identical with t_0 (the inverse of the seismicity rate). We can see that an earthquake with magnitude M is equivalent with $e^{\beta M}$ fundamental earthquakes (with zero magnitude). If we use the magnitude M instead of the natural magnitude βM, we get $t_r = (t_0/\beta)e^{\beta M}$. Also, we note that the number of fundamental earthquakes with zero magnitude $\Delta N(M = 0)/\beta \Delta M$ per natural-magnitude unit is N_0 (or βN_0 in magnitude units), *i.e.* it is the number of total earthquakes. This makes N_0 a fitting parameter. Since T determines the statistical ensemble, it remains that t_0 should be viewed as a fitting parameter.

The discussion given above shows that the fitting parameters in the magnitude distribution are t_0 and β, *i.e.* we should use the formula

$$\frac{\Delta N}{T\Delta M} = \frac{\beta}{t_0} e^{-\beta M} \tag{2.11}$$

in empirical studies.

As regards the recurrence time given by equation (2.10) we note that, by its definition $(T/(\Delta N/\beta\Delta M)$, it is a mean recurrence time; at the same time, it is given by M $(t_r = t_0 e^{\beta M})$, which is a statistical variable. This means that t_r has a dispersion (root mean square deviation, standard deviation) $\delta t_r = t_r$, given by the dispersion $\delta M = \left(\overline{\Delta M^2}\right)^{1/2} = 1/\beta$ (where $\Delta M = M - \overline{M}$). We may use an error $\delta_e M = \left(\sqrt{\overline{M^2}} - \overline{M}\right)/\overline{M} = \sqrt{2} - 1$ and $\delta_e t_r = \sqrt{2} - 1$. (We note that $\overline{M} = 1/\beta$, $\overline{M^2} = 2/\beta^2$ from the magnitude distribution given by equation (2.9)).

Similarly, from equation (2.8) we get the excedence rate (the so-called recurrence law, or the cumulative distribution), which gives the number N (denoted also N_{ex}) of earthquakes with magnitude greater than M. The corresponding probability is readily obtained from (2.8) as $P_{ex} = e^{-\beta M}$, such that the excedence rate can be written as

$$\ln N = \ln N_0 - \beta M \ . \tag{2.12}$$

As discussed above, it is convenient to use this logarithmic formula as

$$\ln (N/T) = -\ln t_0 - \beta M \ , \tag{2.13}$$

where t_0 and β are fitting parameters.

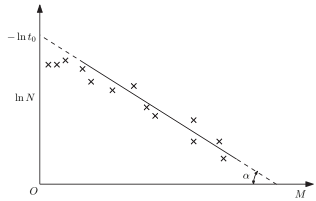


Figure 2.2: Typical empirical cumulative number of earthquakes N fitted by the standard Gutenberg-Richter law (straight line with the slope $-\beta$, $\beta = \tan\alpha$ and the intercept $-\ln t_0$).

The distributions given above are the standard Gutenberg-Richter statistical distributions (equations (2.8) and (2.13)). They assume that the earthquakes are independent statistical events. We call these distributions the distributions of the background (or regular) seismic activity.

2.3 Empirical studies

The cumulative distribution given by equation (2.13) is used to fit the empirical data. The fitting parameters are t_0 and β and the variable is the magnitude M.

First, the knowledge of the magnitude may raise problems. The current procedures relate the magnitude M to the energy of the earthquake and the magnitude of the seismic moment; the later is related to the amplitude (and the frequency) of the seismic waves, measured by seismographs; various corrections are applied. These procedures employ various qualitative estimations, more or less arbitrary, or, at least, of a very particular nature, and are not fully provided by the public domain; it is claimed that they are included in numerical codes, provided by various agencies. The relation energy-magnitude is $E/E_0 = e^{bM}$, where $b = 3.45$. In powers of ten $E/E_0 = 10^{\frac{3}{2}M}$. Since the significant figure in the estimated energy is a large exponent of 10, it is reasonable to assume that a unit error may appear in this exponent, such that the variation of M is $2/3 = 0.66$; this may imply an error $\delta_e M = 0.33$. Such an error may be a serious source of uncertainty.

The empirical number of earthquakes ΔN with the magnitude in the range $(M,\ M+\Delta M)$ is counted by using a step ΔM. In principle, this step should be as small as possible, but for very small ΔM there could be no earthquake, especially for larger magnitudes. The theoretical distribution (*e.g.*, the cumulative distribution $e^{-\beta M}$) is a continuous distribution, while the empirical distribution is a discrete distribution. This is an important source of errors. Also, it is worth noting that the parameter of empirical seismicity rate t_0 (T/N_0) is very different from the fitting parameter t_0 in equation (2.13), precisely because of the difference between a continuous distribution and a discrete distribution and the region of the vanishing magnitudes. In particular, the empirical total number of earthquakes $N_0 = T/t_0$ is very different from the number of zero-magnitude earthquakes (the fundamental earthquakes) $N_0 = \Delta N(M = 0)/\beta\Delta M$, as discussed above. Since the latter is unknown, we view it as a fitting parameter, so t_0 in $N_0 = T/t_0$ is a fitting parameter.

In the fitting procedures we should consider all the earthquakes which occurred in a given seismic region in a long time interval T. This number should be as large as possible, and, consequently, the time interval T should be as long as possible, in order to have a meaning for the probabilities, the statistical variables and the statistical distributions. We say that this large set of earthquakes form a statistical ensemble. (All that we measure in statistical ensembles are mean values, and mean values of the deviations of mean values, known as fluctuations; for relevant results we need small fluctuations, which, presumably, are associated with large ensembles).

A statistical ensemble should be reproducible, *i.e.* we should be able to prepare it in the same conditions many times (or to have many copies of it), such that the measurements made upon it are reproducible. This is impossible with the sets of earthquakes, because we cannot reproduce the conditions of a certain region and a certain period of time T; these conditions may change in time. Also, they vary from region to region; for instance, a variation in the conditions of the statistical ensemble may appear after a large earthquake. A successive number of updates may be considered repetitions in identical conditions of the same statistical ensemble.

In addition, a statistical ensemble should fulfil the condition of the null hypothesis, *i.e.* we should not suspect the existence of a certain, unknown, cause which may favour, or disfavour (bias) certain values of the magnitudes.

Finally, in a statistical ensemble all the earthquakes should be included, in principle, especially those with magnitudes down to zero, which are the most numerous. This is impossible, at least for the fact that the sensitivity of our instruments (seismographs) is limited. Consequently, in empirical studies a lower-magnitude cutoff is always employed, *e.g.* $M \geq 2$ or $M \geq 3$; this cutoff is called the completeness magnitudes of the seismological catalogs. The existence of a lower-magnitude cutoff may induce a serious discussion about the validity of the fit.

Also, in this context, the inclusion of the great earthquakes in a statistical ensemble raises problems, since such events are rare.

In view of all these conditions, practically any empirical realization of a statistical ensemble for the seismic activity is, more or less, a poor copy. Consequently, such a realization may not exhibit fully all the expected features of the statistical ensemble.

Statistical ensembles have been introduced in Statistical Physics by Gibbs, as an instrument of analyzing the thermodynamical ensembles in equilibrium. An important particularity of the thermodynamical ensembles is their so-called extensivity.[7] This means that a statistical ensemble is a collection of a large number N of identical independent statistical sub-ensembles. Although the seismic events characterized by the magnitude M may be viewed as independent events, they are not identical statistical sub-ensembles. The seismic activity is a single statistical process, not a collection of many statistical sub-ensembles. This is an important difference between thermodynamical ensembles and the statistical ensembles of the seismic activity. In particular, the statistical ensemble of the seismic activity should exhibit fluctuations in equilibrium, while the fluctuations of the thermodynamical ensembles are vanishing in the limit of large N (this is sometimes called the central limit theorem). However, we show in this book that the seismic activity does not exhibit fluctuations (at least for Vrancea), which indicates that the seismic activity is not in equilibrium.

[7] J. W. Gibbs, *Elementary Principles in Statistical Mechanics*, Scribner's sons, NY (1902).

An analysis of a large set of global earthquakes with $5.8 < M < 7.3$ ($\Delta M = 0.1$) indicates $\beta = 1.38$ (and $1/t_0 = 10^{5.5}$ per year), corresponding to $r = 0.4$, a value which suggests an intermediate two/three-dimensional focal mechanism.[8] For $r = 1/3$, corresponding to a uniform pointlike focal geometry, we get $\beta = 1.15$. Equations (2.8) and (2.13) have been fitted to a set of 1999 earthquakes with magnitude $M \geq 3$ ($\Delta M = 0.1$), which occurred in Vrancea between $1974 - 2004$ (31 years).[9] The mean values of the fitting parameters are $-\ln t_0 = 9.68$ and $\beta = 1.89$ ($r = 0.54$). A similar fit has been done for a set of 3640 earthquakes with magnitude $M \geq 3$ which occurred in Vrancea during $1981 - 2018$ (38 years).[10] The fitting parameters for this set are $-\ln t_0 = 11.32$ and $\beta = 2.26$ ($r = 0.65$). We note that $\beta = 2.26$ is close to the reference value 2.3. The fitting values given above for Vrancea have an estimated error of approximately 18%. A fit to the excedence law for the Vrancea earthquakes in the period $1981 - 2018$ ($M \geq 3$, $\Delta M = 0.1$) leads to $-\ln t_0 = 10.62$ and $\beta = 2.1$ ($r = 0.61$, error 10%, Fig. 2.1). The data for Vrancea have been taken from the Romanian Earthquake Catalog, 2018. It is accepted that the mean magnitude error in this catalog is $\Delta M = 0.1$ and the completeness magnitude, *i.e.* the magnitude below which the recordings are not reliable, is $M = 2$.

The statistical analysis gives a generic image of a collective, global earthquake focal region (a distribution of foci). Particularly interesting is the parameter r, which is related to the reciprocal of the (average) number of effective dimensions of the focal region and the rate of energy accumulation. The value $r = 0.54$ (Vrancea, period $1974 - 2004$) indicates a (quasi-) two-dimensional geometry of the focal region in Vrancea, while the more recent value $r = 0.65$ for the same region suggests an evolution of this (average) geometry towards one dimension. At the same time, we note an increase of the seismicity rate $1/t_0$ in the recent period in Vrancea. The increase of the

[8] K. E. Bullen, *An Introduction to the Theory of Seismology*, Cambridge University Press, London (1963).

[9] B. F. Apostol, "A Model of Seismic Focus and Related Statistical Distributions of Earthquakes", Phys. Lett. **A357** 462 (2006); "A Model of Seismic Focus and Related Statistical Distributions of Earthquakes", Roum. Reps. Phys. **58** 583 (2006).

[10] B. F. Apostol, *Seismology*, Nova, NY (2020).

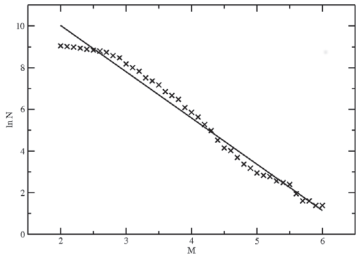


Figure 2.3: Cumulative law $\ln N = \ln N_0 - \beta M$ fitted to the Vrancea earthquakes, $1980-2019$, $M \geq 2$, $\Delta M = 0.1$, for $\beta = 2.22$ and $-\ln t_0 = 10.77$.

geometrical parameter r determines an increase of the parameter β, which dominates the mean recurrence time. For instance, the accumulation time (recurrence time) for magnitude $M = 7$ ($t_r = t_0 e^{\beta M}$) is increased from $t_r \simeq 34.9$ years (period $1974-2004$) to $t_r \simeq 90$ years period $1981 - 2018$). If we use the values $-\ln t = 10.62$ and $\beta = 2.1$ obtained from the excedence law in the period $1981 - 2018$ we get a recurrence time $t_r = 59$ years for $M = 7$. This large variability indicates the great sensitivity of the statistical analysis to the data set. In particular, for any fixed M we may view the exponential $Me^{-M\beta}$ as a distribution of the parameter β, which indicates an error $\simeq 0.41$ in determining this parameter.

2.4 Vrancea background seismic activity

A typical set of empirical cumulative number of earthquakes N (all earthquakes with magnitude greater than, or equal to M), fitted by the standard Gutenberg-Richter distribution $\ln(N/T) = -\ln t_0 - \beta M$, is shown in Fig. 2.2. We can see that the empirical data are more or less scattered around a descending straight line, with the slope $-\beta$ and the intercept $-\ln t_0$. The deviation from a straight line may be larger for large magnitudes, where the data are poor. The empirical data show, typically, a flattened straight line at small magnitudes.

This is called the roll-off effect.[11] Usually, it is assigned to an insufficient realization of the statistical ensemble. We show in this book that the roll-off effect may arise, at least partially, by earthquake correlations. In empirical studies these small-magnitude data are left aside, by using a lower-magnitude cutoff. The remaining data preserve their straight-line character, more or less, such that we may accept that we have a more or less reliable determination of the slope parameter β. However, the parameter t_0 is affected by errors. These parameters, determined in this way, correspond to the background seismic activity of regular earthquakes, *i.e.* they do not include the effect of the correlations. The Gutenberg-Richter distribution employed for the determination of these parameters is called the standard Gutenberg-Richter distribution. We show in this book that the correlations modify the Gutenberg-Richter distribution, especially in the region of small magnitudes, but they preserve the slope of the straight line for moderate and larger magnitudes (where the cumulative distribution is shifted upwards by $\ln 2$).

3561 Vrancea earthquakes with magnitude $M \geq 3$ ($\Delta M = 0.1$), period $1980 - 2019$ ($T = 40$ years), have been fitted to the standard cumulative Gutenberg-Richter distribution $\ln(N/T) = -\ln t_0 - \beta M$, with the fitting parameters $-\ln t_0 = 11.81$, $\beta = 2.44$. These parameters are similar to those given previously, with a slight increase of β. For the same period 8455 earthquakes with magnitude $M \geq 2$ have been analyzed by using the same law, with parameters $-\ln t_0 = 10.77$, $\beta = 2.22$. (The error of these fits are 10%, given by the standard deviation $\left[\sum_{i=1}^{N}(d_i - f_i)^2/N\right]^{1/2}$, where d_i are data and f_i is the function).[12] All the analzyed earthquakes occurred in Vrancea region with $45^\circ - 46^\circ$ latitude and $26^\circ - 27^\circ$ longitude, at various depths (between,

[11] J. D. Pelletier, "Spring-block models of seismicity: review and analysis of a structurally heterogeneous model coupled to the viscous asthenosphere", in J. B. Rundle, D. Turcote, and W. Klein, eds., *Geocomplexity and the Physics of Earthquakes*, vol. **120**, Am. Geophys. Union, NY (2000); P. Bhattacharya, C. K. Chakrabarti, Kamal and K. D. Samanta, "Fractal models of earthquake dynamics", in H. G. Schuster, ed., *Reviews of Nonlinear Dynamics and Complexity*, Wiley, NY (2009), p.107.

[12] $\left[\sum_{i=1}^{N}(d_i - f_i)^2/N\right]^{1/2}$ is the root mean square (*rms*) error; $\left[\sum_{i=1}^{N}\left[(d_i - f_i)/f_i\right]^2/N\right]^{1/2}$ is the root mean square relative error.

approximately, $5km$ to $150km$). It is worth noting that the data with $M \geq 2$ exhibit the roll-off effect (Fig. 2.3).

In all subsequent calculations included in this book we use the background seismicity parameters $-\ln t_0 = 11.32$ and $\beta = 2.26$ ($r = 0.65$) for Vrancea, as resulted form the analysis of 3640 earthquakes, with magnitude $M \geq 3$, over the period $1981 - 2018$ (with 18% error).

The proper fitting to the data should be carried out by using the correlated-earthquake cumulative distribution, which reads $\ln(N/T) = -\ln t_0 + \ln 2 - \ln\left(1 + e^{\beta M}\right)$ (given in the Correlations. Foreshocks. Short-term Prediction chapter). Unfortunately, we have not enough data for small magnitudes, for getting relevant fitting parameters. For instance, we get for the above data $-\ln t_0 = 10.08$, which is just by $\ln 2$ smaller than 10.77 found above with the standard distribution. Smaller magnitudes remove $\ln 2$ in the exact formula. On the other hand, the parameter β may be slightly reduced by the correlation-modified formula. We conclude that the cumulative standard Gutenberg-Richter distribution, employed for data sets with a lower-magnitude cutoff, leads to an underestimate of the parameter t_0 by a factor 2; also, an overestimate of β may be obtained from the standard formula. We note that the correlation-modified formula may have a little weight in the whole ensemble of earthquakes, in comparison to the regular earthquakes.

The correlations which modify the standard Gutenberg-Richter formula occur in foreshock-aftershock sequences associated mainly to high-magnitude main shocks, which may have a large productivity of accompanying earthquakes. We expect the proportion of these sequences to the number of regular earthquakes to be small. It can be obtained as a fitting parameter in the combined formula of the two distributions with their own weights. This is why we expect the standard Gutenberg-Richter distribution to give results not far away from the real ones.

By comparing the results described above, we can see that there exists a change in the parameters of the background seismicity. This change is due to the differences in the data sets, on one hand, but, on the other hand, a change in the geological and the seismicity conditions of the region cannot be excluded. Updating from time to time this analysis may give an insight into the expected estimates of the recurrence times. Also, the basic parameters t_0 and β of the background seismicity are useful in the statistical analysis of other variables.

Task #1 of Practical Seismology is to determine the background seismicity parameters t_0 and β of a given seismic region, like Vrancea, and a given (long) period of time, by using the cumulative standard Gutenberg-Richter for data sets with a lower-magnitude cutoff ($M \geq 3$). The main result is the estimation of the mean recurrence time $t_r = t_0 e^{\beta M}$.

2.5 Physical mechanism

We emphasize the fundamental role played by the E_0-events, which we call fundamental earthquakes (or fundamental seismic events). At any moment of time we may have an E_0-event in a time cell of duration t_0. This energy may be released immediately after this time, by relaxation, and we have not an earthquake, because the energy E_0 corresponds to magnitude $M = 0$ (according to the definition of the magnitude $E/E_0 = e^{bM}$). It follows that in any measurement of time we need a cutoff time t_0; and in any measurement of energy we need a cutoff energy E_0. On the other hand, energies E_0 may not be released. They may accumulate during a time t, consisting of a number of time intervals t_0, given by $t/t_0 = e^{\beta M}$. This energy may be released at time t, as an earthquake with magnitude M. It follows that earthquakes are made of a number of fundamental E_0-events. The seismic activity is the process of accumulation and release of energy, due to the fundamental earthquakes.

Therefore, in a time interval t we may have an E_0-event or an earthquake with energy $E > E_0$. This is a sure event, with probability 1. We have immediately the equation (2.5) and the time probability of an earthquake with the accumulation time t, as given by equation (2.6). By using this probability, the accumulation law and the definition of the magnitude we derive immediately the (standard) Gutenberg-Richter distributions.

Particularly interesting is the Gutenberg-Richter excedence law, which tells that the logarithm of the number of earthquakes with magnitude greater than M is a linear function of M with a slope β and the intercept $-\ln t_0$ (equation (2.13)). This theoretical prediction should be checked against empirical data.

Practically all data exhibit a straigt-line pattern, according to the theoretical prediction, except for a flat region for small magnitudes (the roll-off effect, like in Fig. 2.3). Obviously, the fitting of the theoretical (standard) equation to data should avoid this flat region. The cutoff magnitude can easily be read from data, such that there is no need to introduce a cutoff-magnitude parameter M_c, which, in any case, does not enter the fitting formula. In spite of this obvious situation, there exist many attempts of using a small-magnitude cutoff (as well as an upper-magnitude cutoff) in fitting data. Generally speaking, any set of data may be fitted with any function. As such, the fits have no relevance. They are relevant only when we have a theory which gives a meaning to the parameters entering the fitting formula. In the attempts mentioned above there is no such theory for an M_c (or an upper-magnitude cutoff).

Actually, the Gutenberg-Richter distribution is slightly erroneous. We call it the standard Gutenberg-Richter distribution. We have shown that it is modified by correlations. When using this correlation-modified distribution the small-magnitude flat region may be included in fitting data. The linear part of the standard Gutenberg-Richter distribution remains unaffected by correlations, so the parameter β is, practically, not affected. The intercept $-\ln t_0$ is affected. However, we expect the weight of the correlated earthquakes to be small in comparison with the regular earthquakes.

3 Aftershocks. Next Earthquake

3.1 Conditional probabilities. Omori law

Any earthquake is preceded and followed by other earthquakes, all occurring approximately in the same region and within a reasonably short time interval. In some cases, a big earthquake may be preceded by many earthquakes and followed by many others. The number of the preceding earthquakes is, in general, different from the number of the succeeding earthquakes. Some of these accompanying earthquakes may be regular (background) earthquakes. Some others may be connected between them and to the main earthquake, at least, simply, because they are associated in time and space, or because we chose them according to some constraints. We say that these earthquakes are correlated earthquakes. We call the precursory earthquakes foreshocks; they precede the main earthquake. We call aftershocks the subsequent earthquakes, which follow, succeed the main earthquake. The main earthquake is called in this case a main shock; the foreshocks and the aftershocks are smaller in magnitude than the main shock. Some foreshocks or aftershocks are regular, some others may be correlated, either by physical causes, or by physical constraints, or simply by the constraints implied by their definition.

The correlations may have known physical causes, in which case they are called dynamical correlations, or may have unknown physical causes, in which case we call them purely statistical correlations; by unknown causes we understand causes which are not included in the theoretical models employed to analyze the earthquakes, or implied by their definition. Usually, the correlations are present in statistical distributions, such that all such correlations are in fact statistical correlations. However, there exist also purely deterministic correlations, which affect the parameters of the earthquakes. In general, since the physical causes (known or unknown) are symmetrical with respect to the time reversal, we may expect that the number of foreshocks and their various distributions are identical with the number and the distributions of the aftershocks.[1] However, in empirical studies an appreciable asymmetry often exists, which may not arise only from an insufficiency of the empirical realization of the statistical ensembles.[2] For instance, there often exists a significant increase in the number of small-magnitude aftershocks with respect to the background seismic activity, in contrast with the number of foreshocks.[3] This circumstance may arise not only from the insufficiency of the realization of the statistical ensemble, but it may also arise from the fact that after a main shock the seismic conditions of the region have changed, sometimes appreciably, leading to many succeeding small earthquakes. In this case, it may seem more appropriate to view the main shock and these small-magnitude aftershocks as a single (collective) seismic event.

As long as the aftershocks, irrespective of being regular or correlated earthquakes, are referred to the main shock they are described by conditional probabilities. Similarly, the foreshocks are described by conditional probabilities. The most interesting case is the conditional probability in the time variable.

We know that the probability density for an earthquake to occur at time t is

$$P(t) = \frac{t_0}{t^2} \ . \tag{3.1}$$

Let us assume that a main shock occurs at time t_{ms}, followed by an aftershock occurring at time $t_{ms} + \tau$. The above probability density

[1] D. Vere-Jones, "A note on the statistical interpretation of Bath's law", Bull. Seismol. Soc. Amer. **59** 1535 (1969).

[2] T. Utsu, "Aftershocks and earthquake statistics (I,II): Source parameters which characterize an aftershock sequence and their interrelations", J. Fac. Sci. Hokkaido Univ., Ser. VII, **2** 129, 196 (1969); "Statistical features of seismicity", International Geophysics **81** Part A, 719 (2002).

[3] L. Gulia, A. P. Rinaldi, T. Tormann, G. Vannucci, B. Enescu and S. Wiemer, "The effect of a mainshock on the size distribution of the aftershocks", Geophys. Res. Lett. **45** 13277 (2018).

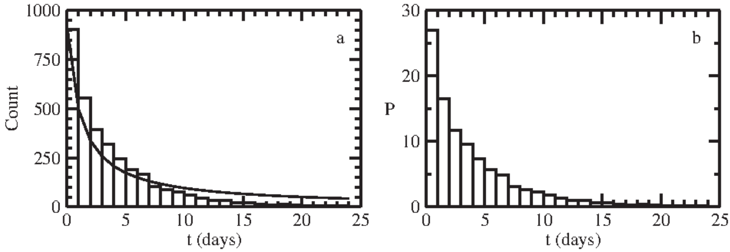


Figure 3.1: Next-earthquake distribution of seismic events in Vrancea 1981 – 2018 (3640 events, $M \geq 3$, panel a) and probabilities $P = P(t)$ (in %, panel b). The fitting curve in panel a is $1066.45/(1.15+t)$ (coefficient of determination $\simeq 0.96$).

reads

$$P(t_{ms}+\tau) \simeq \frac{t_0}{t_{ms}^2}\frac{t_{ms}/2}{t_{ms}/2+\tau} \tag{3.2}$$

for $\tau \ll t_{ms}$. We can see that this equality shows that a combined event, consisting of a main shock and an aftershock, occurs with a probability $\sim P(t_m+\tau)$ which is the product of the probability $\sim t_0/t_{ms}^2$ of the main shock and the probability $\sim 1/(t_{ms}/2+\tau)$ of the aftershock. This equality is precisely the definiton of the conditional probability $\sim 1/(t_{ms}/2+\tau)$.[4]

We denote by t_c the time $t_{ms}/2$ and normalize the aftershock probability, whose density is denoted by P_0, in the interval $0 < \tau < t_c$; this is an arbitrary normalization. We get

$$P_0(\tau) = \frac{1}{\ln 2}\frac{1}{t_c+\tau}\ ; \tag{3.3}$$

this law is valid for $\tau \ll t_c$. The parameter t_c is viewed as a fitting parameter. Since we assume a foreshock-aftershock symmetry we extend this law to the foreshocks, where τ is measured from the main

[4]T. Bayes, "An essay towards solving a problem in the doctrine of chance" (communicated by R. Price in a letter to J. Canton), Phil. Trans. Roy. Soc. London **53** 370 (1763).

shock in the past. This empirical law is known as Omori's law.[5] It may also be derived for a self-replication process which generates aftershocks (and foreshocks).[6] We show in this book that the law holds for both the regular-earthquake probability given by equation (3.1) and the correlated-earthquake probability. We note that Omori's law does not reflect correlations, generated by physical causes. It reflects the simple fact that after a main shock follows an aftershock, or before a main shock there exists a foreshock. It gives the probability of occurring an accompanying seismic event with the condition of the existence of a main shock (it is a conditional probability). This is a (statistical) correlation arising from the definition of the events

We note that, according to Omori's law, the probability of the aftershocks decreases in time, because high-magnitude earthquakes, which are favoured for longer times, are less numerous. We expect to have many small-magnitude earthquakes in the vicinity of the main shock.

3.2 Next-earthquake distribution

From the derivation of Omori's law we see that there is no particular distinction between the main shock and the aftershocks, or foreshocks. The law may be applied to any pair of earthquakes separated by time τ. Such a time distribution is called the next-earthquake (or inter-event) distribution.

The probability density of N serial events denoted by i and occurring at time t_i can be written as $N^{-1}\sum_i \delta(t_i - t)$. Similarly, the pair distribution of nearest-neighbours separated by time t is given

$$P(t) = \frac{dN}{Nd\tau} = \frac{1}{N}\sum_i \delta(t_{i+1} - t_i - t) \ . \tag{3.4}$$

This function is also known as the next-earthquake distribution, recurrence, or waiting-time distribution, or inter-event time distribution.

[5] F. Omori, "On the after-shocks of earthquakes", J. Coll. Sci. Imper. Univ. Tokyo **7** 111 (1894).

[6] B. F. Apostol, "Euler's transform and a generalized Omori's law", Phys. Lett. **A351** 175 (2006).

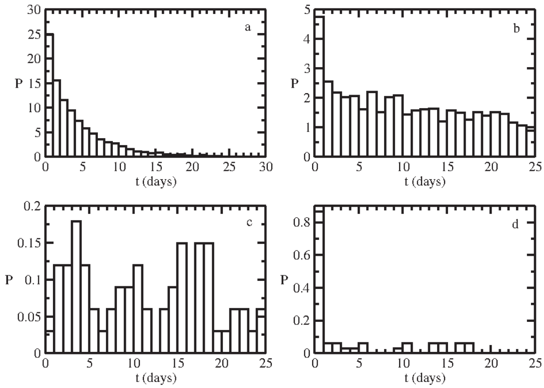


Figure 3.2: Next-earthquake distributions $P = P(M,t)$ (in %) for Vrancea 1981 – 2018 for $M_0 \geq 3$ and $3 \leq M < 4$ (panel a), $4 \leq M < 5$ (panel b), $5 \leq M < 6$ (panel c) and $6 \leq M$ (panel d).

The earthquakes that occurred in Vrancea between 1981 – 2018 with $M \geq 3$ (Romanian Earthquake Catalog, 2018) are distributed in Fig. 3.1 on the inter-event time (panel a); the corresponding probabilities $P(t)$ (in %) are shown in Fig. 3.1 panel b (time is measured in days on the abscissa).[7] The rate of occurrence per day of the next earthquake follows a power-law time dependence (Omori-type law) over a time window of about 25 days. The distribution is fitted with the law $a/(b+t)$, where $a = 1066.45$, $b = 1.15$ and t denotes the time (coefficient of determination $R = 0.96$).[8] The mean time ($\overline{t}$) for $P(t)$ is $\simeq 5.89$ days, and the variance is $\sigma = 9.55$ days ($\sigma = \left(\overline{t^2} - \overline{t}^2\right)^{1/2}$). We note the presence of the cutoff time b. To check whether this behavior is biased by the aftershock sequences of the strongest seisms of the investigated time period (four earthquakes with magnitude ≥ 6.0), it

[7]B. F. Apostol and L. C. Cune, "Short-term seismic activity in Vrancea. Inter-event time distributions", Ann. Geophys. **63** SE328 (2020); doi: 10.4401/ag-8366.

[8]The coefficient of determination R is defined by $R^2 = 1 - \sum_i (d_i - f_i)^2 / \sum_i (d_i - \overline{d})^2$, where d_i denote the data, f_i denote the fit and $\overline{d}$ is the mean value of the data.

was considered also two shorter time intervals: $1991-2018$, avoiding the aftershock sequences of three events with $M > 6.0$ (occurred on August 30, 1986, $M = 7.1$, May 30, 1990, $M = 6.9$, and May 31, 1990, $M = 6.4$), and $2005-2018$, when no earthquake larger than magnitude 5.6 occurred. The results show very similar next-earthquake probabilities of occurrence, in all three cases ($a = 881.85$, $b = 1.25$, $R = 0.94$ for $1991-2018$ and $a = 492.6$, $b = 1.16$, $R = 0.93$ for $2005-2018$). We note that the cutoff time b is much smaller than the recurrence time t_{ms} of a main shock, which occurs in the original derivation of Omori's law, since we included all the earthquakes, not only main shocks with high magnitudes (next-earthquake distribution).

From a practical standpoint a relevant question in short-term earthquake forecasting seems to be "what happens next?". Let us assume that an earthquake occurs at time t_0 and the next one occurs at some time t measured with respect to t_0. We can define a distribution $P(t)$ of these next earthquakes, and determine it from a set of relevant statistical data. Once determined, it can be used for estimating the time probability of occurrence of the next earthquake, based on the principle "what happened will happen again". For instance, from Fig. 3.1, panel b, we can say that the probability for an earthquake with magnitude $M \geq 3$ to occur in the next day after an earthquake with magnitude $M \geq 3$ has occurred is $\simeq 27\%$. Let the earthquakes be labelled by some generic parameter x, like magnitude, location, depth, etc. Then, we may distribute the next earthquakes with respect to x, and introduce the time probability distribution $P(x,t)$ of the next earthquake characterized by parameter x occurring at time t. Another distribution $P(x,t \mid x_0)$ may also be introduced with respect to an earthquake labelled by parameter x_0, which is a conditional probability. The procedure may obviously be extended, by introducing, similarly, the probability distributions $P(x,t \mid x_{01},x_{02},...)$, or $P(x_1,x_2,...,t \mid x_{01},x_{02},...)$, which resemble the hierarchy of n-point correlation functions. Characteristic scale time or size, or correlation range could be identified from the statistical analysis of such functions, providing the statistical set of data is large enough, which may shed light on the statistical patterns of a seismic activity. The statistics is rather poor, in general, precisely for those ranges of x where the estimation of the seismic hazard and risk is most interesting, like, for instance, for x corresponding to high values of magnitude M. The generic parameter x in the analysis of the inter-event time distributions described here is the magnitude M.

The probabilities $P(M,t)$ for $M_0 \geq 3$ for Vrancea during $1981-2018$ are shown in Fig. 3.2 for $3 \leq M \leq 4$, $4 \leq M \leq 5$, $5 \leq M \leq 6$, and $6 \leq M$ (panels a, b, c and d, respectively). $P(M,t)$ is the probability for an earthquake with magnitude M to occur after a time t since the occurrence of an earthquake with magnitude M_0. First, we note that the inter-event distributions $P(M,t)$ for Vrancea exhibit a characteristic decrease in time, with the highest probability of next-earthquake occurrence in the same day as the reference earthquake, at least for small magnitudes ($M < 5$). Then, we note the decreasing maximum values of these probabilities ~ 22.7 for $3 \leq M < 4$, ~ 2.75 for $4 \leq M < 5$, while the probability $P(M,t)$ vanishes practically for $M > 5$. Also, it is worth noting that $P(t)$ and $P(3 \leq M < 4, t)$ are similar, obeying Omori-type power laws, at least for short times, while the distributions become gradually irregular, exhibiting large fluctuations on increasing magnitude above $M = 4-5$. The statistics becomes poor for higher magnitude ($M > 5$), as expected. A correlation time of $20-25$ days can be estimated, after which the probabilities decrease appreciably, as well as a size correlation of $M = 4-5$, above which the distributions acquire very small values, and are very irregular. The null hypothesis was tested on these distributions, by comparing the results of the first half of data with those derived from the second half of data.

Task #2 of Practical Seismology is to produce and use Figures like Figs. 3.1, 3.2 shown above. The probability of the next seismic event can be read on such Figures after each earthquake.

4 Correlations. Foreshocks. Short-term Prediction

4.1 Time-magnitude correlations

We know that the law of energy accumulation in the focus is $t/t_0 = (E/E_0)^r$, where t is the accumulation time, r is the focus parameter and t_0 and E_0 are time and energy thresholds.

Let the amount of energy E accumulated in time t be released by two successive earthquakes with energies $E_{1,2}$, such as $E = E_1 + E_2$. According to the accumulation law

$$\begin{aligned} t/t_0 = (E/E_0)^r = (E_1/E_0 + E_2/E_0)^r < \\ < (E_1/E_0)^r + (E_2/E_0)^r = t_1/t_0 + t_2/t_0 \ , \end{aligned} \tag{4.1}$$

where $t_{1,2}$ are the accumulation times for the energies $E_{1,2}$. We can see that the time corresponding to the pair energy is shorter than the sum of the independent accumulation times of the members of the pair. This is a type of deterministic time-magnitude (time-energy) correlations, arising from the non-linearity of the accumulation law. These correlations are not statistical (though they may affect the statistical distributions); they occur just by a partition of energy among two (or more) earthquakes, as a consequence of the seismicity conditions. They may appear in the foreshock-main shock-aftershock sequences.

According to the above equation, the time interval τ between the two successive earthquakes is given by

$$\tau = t_1 \left[(1 + E_2/E_1)^r - 1 \right] \ , \tag{4.2}$$

where $t = t_1 + \tau$ and $t_1 = t_0 e^{\beta M_1}$ is the accumulation time of the earthquake with magnitude M_1. If we introduce the magnitudes $M_{1,2}$ in equation (4.2) (by $E/E_0 = e^{bM}$), we get

$$\tau = t_1 \left[\left(1 + e^{-bm}\right)^r - 1 \right] \ , \tag{4.3}$$

where $m = M_1 - M_2$. This equation relates the time τ to the magnitude difference m.

From the above equations we get the magnitude

$$M_2 = M_1 + \tfrac{1}{b} \ln \left[(1 + \tau/t_1)^{1/r} - 1 \right] \ , \tag{4.4}$$

or

$$M_2 \simeq \frac{1}{b} \ln \frac{\tau}{\tau_0} \ , \ \tau_0 = r t_0 e^{-(1-r)bM_1} \tag{4.5}$$

for $\tau > \tau_0$ ($M_2 > 0$).[1] The magnitude M_2 is plotted *vs* $\theta = \tau/\tau_0$ in Fig. 4.1 for $b = 3/2$ (decimal logarithms in equation (4.5)).

We can apply these equations to a foreshock-main shock-aftershock sequence, where M_1 is the magnitude of the main shock and M_2 is the magnitude of the foreshocks (aftershocks). In this case we need to impose also the condition $M_2 < M_1$, *i.e.* $\tau < t_1(2^r - 1)$ (or $\tau < rt_1$). We can see that after a main shock there is a small quiescence time τ_0, followed by an increase in the aftershock magnitudes;[2] similarly, very close to a main shock there are small-magnitude foreshocks followed by no seismic activity until the main shock. An estimation of τ_0 for $r = 2/3$, $-\ln t_0 = 11.32$ (years, Vrancea) and $M_1 = 7$ ($b = 3.45$) gives $\tau_0 \simeq 10^{-8.53}$ years ($\simeq 0.09s$).

Very likely, after a main shock the seismicity conditions of the region change. The occurrence of the aftershocks is governed by the accumulation law

$$P(\tau) d\tau = \frac{\tau_0}{\tau^2} d\tau \ , \tag{4.6}$$

where τ and τ_0 are given by equation (4.5). The change of the seismicity conditions is incorporated in the parameter τ_0, which replaces

[1] B. F. Apostol, "Correlations and Bath's law", Res. Geophys. **5** 100011 (2021).

[2] Y. Ogata and H. Tsuruoka, "Statistical monitoring of aftershock sequences: a case study of the 2015 M_w7.8 Gorkha, Nepal, earthquake", Earth, Planets and Space **68** 44 10.1186/s40623-016-0410-8 (2016).

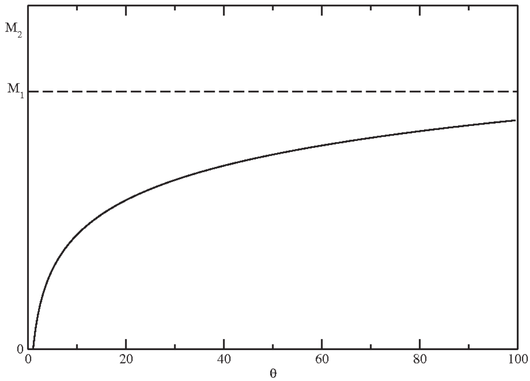


Figure 4.1: Magnitude M_2 plotted *vs* $\theta = \tau/\tau_0$ for $b = 3/2$ (equation (4.5), decimal logarithms).

the parameter t_0 of the regular (background) seismicity. By using $\tau = \tau_0 + t$, we note that we get Omori's law from the above equation (as expected for a conditional probability). From equation (4.6) we get the magnitude distribution of the aftershocks

$$P(M_2) = be^{-bM_2} \ . \tag{4.7}$$

We can see that the magnitude distribution increases, with respect to the background, from β to b for zero magnitudes; this increase lasts up to a magnitude of the order $M_{2c} = \ln r/b(r-1)$ (where $P(M_{2c}) = \beta e^{-\beta M_{2c}}$). We may estimate an average relative increase of the order $(b-\beta)/2\beta = (1-r)/2r$, which, for $r = 2/3$, is $1/4$ (25%). The cutoff magnitude is $M_{2c} = 0.36$ (for $r = 2/3$, $b = 3.45$). Such an increasing tendency of the magnitude distribution of the aftershocks, with respect to the background, in the region of the small magnitudes is documented empirically in some cases.[3]

[3]L. Gulia, A. P. Rinaldi, T. Tormann, G. Vannucci, B. Enescu and S. Wiemer, "The effect of a mainshock on the size distribution of the aftershocks", Geophys. Res. Lett. **45** 13277 (2018).

4.2 Physical mechanism

Let us consider a main shock with energy E_1 (magnitude M_1) occurring at time t_1, followed by an aftershock with energy E_2 (magnitude M_2), occurring after a lapse of time τ. We may assume $E_2 \ll E_1$, and write the first two terms in a Taylor series expansion of the accumulation law $t/t_0 = (E_1/E_0 + E_2/E_0)^r$. We get

$$t = t_1 + rt_1 E_2/E_1 \ ; \tag{4.8}$$

indeed, we can check easily

$$rt_1 E_2/E_1 = rt_0 e^{-(1-r)bM_1} e^{bM_2} = \tau_0 e^{bM_2} = \tau \ , \tag{4.9}$$

according to equation (4.5), *i.e.* $t = t_1 + \tau$. Let us introduce the notation

$$\frac{1}{D(E)} = \frac{\partial t}{\partial E} = rt_0 \frac{E^{r-1}}{E_0^r} = rt/E \ , \tag{4.10}$$

such that equation (4.8) can be written as

$$t = t_1 + \frac{1}{D(E_1)} E_2 \ . \tag{4.11}$$

This equation shows that an energy E_1, accumulated in focus, is released at time t_1 (main shock) and the accumulation continues, such that, after time $\tau = E_2/D(E_1)$, an energy E_2 is released. We note that, in general, the accumulation process is not uniform, it is performed with a rate $D(E) = \partial E/\partial t$ which depends on time (or the accumulated energy). After the main shock, the accumulation may continue, approximately with the rate it had at the moment of the main shock, for a short time interval $\tau = E_2/D(E_1)$, after which the corresponding aftershock (with energy E_2) is produced. The correlation consists in the dependence of τ on E_1.

Let us consider now the foreshocks, where the situation is different. Let us assume that at time $t_2 = t_1 - \tau$ an energy E_1 is accumulated. This energy correponds to the accumulation time t_1, *i.e.* $t_1/t_0 = (E_1/E_0)^r$. The accumulation process $t_2/t_0 = (E_1/E_0)^r$, where $t_2 < t_1$, is possible provided r decreases. Therefore, at time t_2 the seismicity conditions are changed, such that we have a smaller parameter r.

At this moment an energy E_2 is released by a foreshock, and the parameter r recovers its background value. At this moment we have $t_2/t_0 = [(E_1 - E_2)/E_0]^r$, where r has its background value. We write

$$t_1 = t_0(E_1/E_0)^r = t_0(E_1/E_0 - E_2/E_0 + E_2/E_0)^r = \\ = t_2 + \tfrac{1}{D(E_1-E_2)}E_2 \ ; \tag{4.12}$$

we can see that the accumulation process continues from the foreshock time t_2 with an accumulation rate corresponding to the energy $E_1 - E_2$; since $E_2 \ll E_1$ this rate is equal to the rate corresponding to energy E_1. There exists a symmetry between foreshocks and aftershocks as regards their time-magnitude (energy) relationship. However, the physical mechanisms of the foreshocks and the aftershocks are different. Before any foreshock the seismicity conditions change, leading to a decrease of the accumulation parameter r, which recovers its original value immediately after the foreshock. The decrease in the parameter r, indicated for foreshocks by the time-magnitude correlations, reflects the general mechanism of dynamical correlations, valid for both foreshocks and aftershocks.

Both the foreshock and the aftershock mechanisms are described by small relative variations in the accumulation time and the energy. From $t/t_0 = (E/E_0)^r$ we get $\delta t/t = r\delta E/E$; on the other hand, an excess of energy implies a change in the seismicity parameter r, such that $\delta r \cdot \ln(E/E_0) + r\delta E/E = 0$ (from $t/t_0 = const$), or $\delta r/r = -\delta E/bME = -\delta t/\beta Mt$. In these equations E and $E + \delta E$ are the energies of two succeeding events, separated by time $\delta t = \tau$, and t is the occurrence time of the event with energy E and magnitude M. For an aftershock $\delta E < 0$ and $\tau < 0$, such that $\delta r/r = \tau/\beta Mt > 0$, and we can see that r increases. For a foreshock $\delta E > 0$, $\tau > 0$ and $\delta r/r = -\tau/\beta Mt < 0$, such that the parameter r decreases. It is worth noting that $\delta E = E\left(e^{b\delta M} - 1\right)$ cannot be written as $\delta E = bE\delta M$ for realistic values of the magnitude (because $b\delta M$ is not small).

4.3 Correlated Gutenberg-Richter distributions

In general, two or more earthquakes may appear as being associated in time and space with, or without, a mutual interaction between their focal regions. In both cases they form a foreshock-main shock-aftershock sequence which exhibits correlations. The correlations which appear as a consequence of an interaction imply an energy transfer (exchange) between the focal regions (*e.g.*, a static stress). These correlations may be called dynamical (or "causal") correlations. Other correlations may appear without this interaction. For instance, an earthquake may produce changes in the neighbourhood of its focal region (adjacent regions), and these changes may influence the occurrence of another earthquake. Similarly, an associated seismic activity may be triggered by a "dynamic stress", not a static one.[4] The correlations which appear in the statistical distributions are statistical correlations. In this sense, the dynamical correlations are statistical correlations. Other statistical correlations may exist, distinct from dynamical correlations. They arise from conditions imposed on the statistical variables. We call them purely statistical correlations.

We know that $t_0/(t_1+t_2)$ is the probability to have an E_0-event ($M=0$) in the interval $(t_0,\, t)$, $t=t_1+t_2$. Let $\int_{t_0}^{t_1} dt' P_1(t';t_2)$ be the probability to have an earthquake with $E>E_0$ in the interval $(t_0,\, t_1)$ and an E_0-event in the interval $(0,\, t_2)$. We must have

$$\frac{t_0}{t_1+t_2}+\int_{t_0}^{t_1} dt' P_1(t';t_2)=\frac{t_0}{t_0+t_2} \tag{4.13}$$

for any fixed t_2; hence, by differentiation, we get the probability

$$P_1(t_1;t_2)=\frac{t_0}{(t_1+t_2)^2} \tag{4.14}$$

to have an earthquake $E>E_0$ at t_1 and an E_0-event in the interval $(0,\, t_2)$. The probability to have an earthquake $E>E_0$ at t_1 and

[4] K. R. Felzer and E.E. Brodsky, "Decay of aftershock density with distance indicates triggering by dynamic stress", Nature **441** 735 (2006).

another earthquake with energy greater then E_0 in the interval $(0,\, t_2)$ is $\int_0^{t_2} dt' P(t_1, t')$; therefore, we must have

$$\frac{t_0}{(t_1+t_2)^2} + \int_0^{t_2} dt' P(t_1, t') = \frac{t_0}{t_1^2} \ , \tag{4.15}$$

which leads immediately to

$$P(t_1, t_2) = \tfrac{2t_0}{(t_1+t_2)^3} \tag{4.16}$$

(normalized with a cutoff time only for one variable, $t_0 < t_1$). We note that this distribution corrresponds to two earthquakes which are mutually conditioned; it is symmetric in $t_{1,2}$. Consequently, they involve correlations generated by physical causes, like a transfer of stress or energy. These are dynamical correlations. Moreover, it is easy to see that the law given above leads to Omori's law, *e.g.* for $t_2 \ll t_1$, as expected for conditional probabilities. The factor $\sim 1/t_1^3$ obtained in this case, instead of the factor $1/t_1^2$, reflects the fact that the earthquake occurring at t_1 is conditioned by the succeeding earthquake, in contrast with the law $\sim 1/t_1^2$, which does not imply such a condition.[5] Also, we note that the correlations appear through the frequency $t_0/(t_1+t_2)$ of the E_0-events in the whole time interval t_1+t_2. This probability is distinct from the probability $(t_0/t_1)(t_0/t_2)$, which corresponds to two independent sets of E_0-events.

Introducing $t_{1,2} = t_0 e^{\beta M_{1,2}}$ in equation (4.16) we get the pair distribution in magnitudes (also called the two-event, bivariate distribution)[6]

$$P(M_1, M_2) = 4\beta^2 \frac{e^{\beta(M_1+M_2)}}{\left(e^{\beta M_1} + e^{\beta M_2}\right)^3} \ . \tag{4.17}$$

We can see that this distribution is different from $P(M_1)P(M_2) = \beta^2 e^{-\beta(M_1+M_2)}$, which indicates that the two events $M_{1,2}$ are correlated.

Let $M_1 = M_2 + m$ and $M_1 > M_2$, $0 < m < M_1$; equation (4.17) becomes

$$P(M_1, M_2) = 4\beta^2 \frac{e^{-\beta M_1} e^{-\beta m}}{\left(1 + e^{-\beta m}\right)^3} \ ; \tag{4.18}$$

[5]B. F. Apostol, *Seismology*, Nova, NY (2020).

[6]B. F. Apostol, "Correlations and Bath's law", Res. Geophys. **5** 100011 (2021).

similarly, for $M_2 > M_1$, $-M_2 < m < 0$ we get

$$P(M_1, M_2) = 4\beta^2 \frac{e^{-\beta M_2} e^{\beta m}}{\left(1 + e^{\beta m}\right)^3} \ . \tag{4.19}$$

It follows that we may write

$$P(M_1, M_2) = 4\beta^2 \frac{e^{-\beta max(M_1, M_2)} e^{-\beta |m|}}{\left(1 + e^{-\beta |m|}\right)^3} \ , \tag{4.20}$$

which highlights the magnitude-difference distribution, with the constraint $\mid m \mid < max(M_1, M_2)$.

Let M_s and M be the magnitudes of the main shock and an accompanying earthquake (foreshock or aftershock), respectively. We define the ordered magnitude difference $m = M_s - M > 0$ for foreshocks and $m = M - M_s < 0$ for aftershocks, $\mid m \mid < M_s$. According to equation (4.20), the distribution of the pair consisting of the main shock and an accompanying event is

$$P(M_s, m) = 4\beta^2 e^{-\beta M_s} \frac{e^{-\beta |m|}}{\left(1 + e^{-\beta |m|}\right)^3} \ , \ \mid m \mid < M_s. \tag{4.21}$$

The exponential $e^{-\beta |m|}$ falls off rapidly to zero for increasing m, so we may neglect it in the denominator in equation (4.21). We are left with the pair distribution

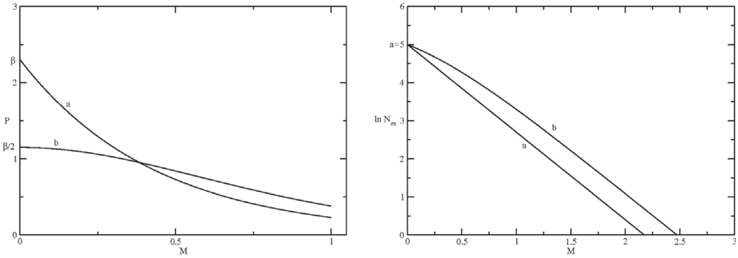
$$P(M_s, m) = \beta^2 e^{-\beta M_s} e^{-\beta |m|} \ , \ \mid m \mid < M_s \ , \ M_s > 0 \tag{4.22}$$

(properly normalized).

If we integrate equation (4.17) with respect to M_2 (and redefine $M_1 = M$), we get the so-called marginal distribution

$$P^c(M) = \beta e^{-\beta M} \frac{2}{\left(1 + e^{-\beta M}\right)^2} \ . \tag{4.23}$$

This distribution differs appreciably from the standard Gutenberg-Richter distribution $\beta e^{-\beta M}$ for $\beta M \ll 1$ and only slightly (by an almost constant factor $\simeq 2$) for moderate and large magnitudes, as



(a) (b)

Figure 4.2: The standard GR distribution $\beta e^{-\beta M}$ (panel (a), curve a) compared to the correlation-modified GR distribution, equation (4.23) (panel (a), curve b) and the standard cumulative GR distribution $\ln N_{ex} = a - \beta M$ (panel (b), curve a) compared to the correlation-modified cumulative GR distribution, equation (4.24) (panel (b), curve b) for $\beta = 2.3$ and an arbitrary value $a = 5$.

shown in Fig. 4.2. We note that the mean magnitude $\overline{M}^c = 2\ln 2/\beta \simeq 1.39/\beta$ is only slightly greater than the mean magnitude $\overline{M} = 1/\beta$ computed with the standard Gutenberg-Richter distribution. The corresponding cumulative distribution for all magnitudes greater than M is

$$P_{ex}^c(M) = e^{-\beta M}\frac{2}{1+e^{-\beta M}} \ . \tag{4.24}$$

This distribution can be written as

$$P_{ex}^c(M) \simeq e^{-\beta M}\frac{1}{1-\frac{1}{2}\beta M} \simeq e^{-\frac{1}{2}\beta M} \tag{4.25}$$

in the limit $\beta M \longrightarrow 0$, which indicates that the slope of the excedence rate

$$\ln P_{ex}^c(M) = \ln\left(\frac{2}{1+e^{-\beta M}}\right) - \beta M \tag{4.26}$$

deviates from $-\beta$, corresponding to the standard Gutenberg-Richter exponential distribution, to $-\frac{1}{2}\beta$ (Fig. 4.2). This roll-off effect[7]

[7] J. D. Pelletier, "Spring-block models of seismicity: review and analysis of a

is attributed usually to an insufficient determination of the small-magnitude data. We can see that it may arise, at least partially, from dynamical correlations. This deviation indicates the presence of dynamical correlations governed by the distribution law $P_c(M) = \frac{1}{2}\beta e^{-\frac{1}{2}\beta M}$ for small magnitudes. It can be shown that the law $\frac{1}{2}\beta e^{-\frac{1}{2}\beta M}$ indicates fluctuations in the number of fundamental earthquakes $e^{\beta M} = t/t_0$ (square root of this number), as arising from mutual interactions of events with various magnitudes.[8] The slope of the modified Gutenberg-Richter distribution given by equation (4.26) is practically the same as the slope $-\beta$ of the standard Gutenberg-Richter distribution for βM greater than $\simeq 2$. For large values of βM the modified distribution given by equation (4.26) is shifted above the standard Gutenberg-Richter distribution by $\simeq \ln 2$. The distributions $P^c(M)$ (equation (4.23)) and $P^c_{ex}(M)$ (equation (4.26)) are correlation-modified Gutenberg-Richter distributions (different from the standard Gutenberg-Richter distributions $\beta e^{\beta M}$ and $e^{-\beta M}$). Such a qualitative behaviour seems to be exhibited by the cumulative distributions of southern California earthquakes recorded between 1945-1985 and 1986-1992.[9] The difference arises mainly in the small-magnitude region $M \lesssim 1$, where the distributions are flattened.

In empirical studies an earthquake may be either correlated, or uncorrelated (regular, independent); let c and s denote the corresponding weights, $c + s = 1$. For a correlated earthquake with magnitude M the cumulative distribution is $P^c_{ex}(M)$ (equation (4.25)), while for a regular earthquake with magnitude M the cumulative distribution is $e^{-\beta M}$. Consequently, we can write the cumulative distribution

$$P_{ex}(M) = c \cdot e^{-\beta M} \frac{2}{1 + e^{-\beta M}} + s \cdot e^{-\beta M} \ . \tag{4.27}$$

The asymptotic behaviour of this (logarithmic) distribution is given

structurally heterogeneous model coupled to the viscous asthenosphere", in J. B. Rundle, D. Turcote and W. Klein, eds., *Geocomplexity and the Physics of Earthquakes*. vol. **120**, Am. Geophys. Union, NY (2000); P. Bhattacharya, C. K. Chakrabarti, Kamal and K. D. Samanta, "Fractal models of earthquake dynamics", in H.G. Schuster, ed., *Reviews of Nolinear Dynamics and Complexity*, Wiley, NY (2009), p.107.

[8] B. F. Apostol, "Correlations and Bath's law", Res. Geophys. **5** 100011 (2021).

[9] L.M. Jones, "Foreshocks, aftershocks and earthquake probabilities: accounting for the Landers earthquake", Bull. Seism. Soc. Am. **84** 892 (1994).

by

$$\ln P_{ex}(M) = \ln\left(\frac{2c}{1+e^{-\beta M}} + s\right) - \beta M =$$

$$= \begin{cases} -\frac{1+s}{2}\beta M \ , \ M \longrightarrow 0 \ , \\ \ln(1+c) - \beta M \ , \ M \longrightarrow \infty \ . \end{cases} \tag{4.28}$$

We can see that the change of the slope from $-(1+s)\beta/2$ to $-\beta$ occurs quickly, over the (narrow) range $\Delta M \simeq \frac{2\ln(1+c)}{\beta c} \simeq 2/\beta$ ($\simeq 0.87$, $\beta = 2.3$). Also, the slope of the roll-off region is increased from $\beta/2$ to $(1+s)\beta/2$. Equation (4.28) can be used to fit the empirical data, small-magnitude region included. Such a fit does not affect appreciably the slope of the standard Gutenberg-Richter distribution.

4.4 Bath's law

Bath's law states that the average difference ΔM between the magnitude of a main shock and the magnitude of its largest aftershock is independent of the magnitude of the main shock.[10] The reference value of the average magnitude difference is $\Delta M = 1.2$. Deviations from this value have been reported.[11] In principle, as a consequence of the foreshock-aftershock symmetry, we may expect a similar law for foreshocks.

Let us apply first the pair distribution given by equation (4.22) to dynamically-correlated earthquakes, by replacing β by $\beta/2$. These earthquake clusters are associated to high-magnitude main shocks, so we may omit the condition $\mid m \mid < M_s$, and let $\mid m \mid$ go to infinity. In this case the statistical correlations are lost; we are left only with

[10] M. Bath, "Lateral inhomogeneities of the upper mantle", Tectonophysics **2** 483 (1965); C. F. Richter, *Elementary Seismology*, Freeman, San Francisco, CA (1958) p. 69.

[11] A. M. Lombardi, "Probability interpretation of "Bath's law", Ann. Geophys. **45** 455 (2002); K. R. Felzer, T. W. Becker, R. E. Abercrombie, G. Ekstrom and J. R. Rice, "Triggering of the 1999 M_w 7.1 Hector Mine earthquake by aftershocks of the 1992 M_w 7.3 Landers earthquake", J. Geophys. Res. **107** 2190 10.1029/2001JB000911 (2002); R. Console, A. M. Lombardi, M. Murru and D. Rhoades, "Bath's law and the self-similarity of earthquakes", *J.* Geophys. Res. **108** 2128 10.1029/2001JB001651 (2003).

the dynamical correlations. The distribution given by equation (4.22) becomes a distribution of two independent events, identified by M_s and m; we may use only the magnitude difference distribution

$$p_c(m) = \frac{1}{4}\beta e^{-\frac{1}{2}\beta|m|} \ , \ -\infty < m < +\infty \ . \tag{4.29}$$

This distribution has a vanishing mean value $\overline{m}$ ($\overline{m} = 0$). The next correction to this mean value, *i.e.* the smallest deviation of m, is the standard deviation

$$\Delta m = \sqrt{\overline{m^2}} = \frac{2\sqrt{2}}{\beta} \ . \tag{4.30}$$

Therefore, we may conclude that the average difference in magnitude between the main shock and its largest aftershock (or foreshock) is given by the standard deviation $\Delta M = \Delta m = 2\sqrt{2}/\beta$. This is Bath's law. The number $2\sqrt{2}/\beta$ does not depend on the magnitude M_s (but it depends on the parameter β, corresponding to various realizations of the statistical ensemble). It is worth noting that Δm given by equation (4.30) implies an averaging (of the squared magnitude differences). Making use of the reference value $\beta = 2.3$ we get $\Delta M = 1.23$, which is Bath's reference value for the magnitude difference. We can check that higher-order moments $\overline{m^{2n}}$, $n = 2, 3, ...$ are larger than $\overline{m^2}$ (for any value of β in the range $1.15 < \beta < 3.45$).

We can estimate the occurrence time τ_B of the Bath earthquake, measured from the occurrence of the main shock, by using $m = 2\sqrt{2}/\beta$ in equation (4.3), where r is replaced by $r/2$ (as for dynamical correlations); we get $\tau_B/t_1 = \frac{1}{2}re^{-2\sqrt{2}/r}$; for $r = 2/3$ we get $\tau_0/t_1 = 5\times10^{-3}$. This time can be taken as an estimate of the duration of the aftershock (foreshock) activity.

The result $\Delta M = 2\sqrt{2}/\beta$ could be tested empirically, although, as it is well known, there exist difficulties. In empirical studies the magnitude difference ΔM is variable, depending on the fitting parameter β, which can be obtained from the statistical analysis of the data. The results may tend to the value $\Delta M = 1.2$ by adjusting the cutoff magnitudes,[12]

[12] A. M. Lombardi, "Probability interpretation of "Bath's law", Ann. Geophys. **45** 455 (2002); R. Console, A. M. Lombardi, M. Murru and D. Rhoades, "Bath's law and the self-similarity of earthquakes", *J.* Geophys. Res. **108** 2128 10.1029/2001JB001651 (2003).

or by choosing particular values of fitting parameters;[13] there are cases where the data exhibit values close to $\Delta M = 1.2$.[14] It seems that values closer to $\Delta M = 1.2$ occur more frequently in the small number of sequences which include high-magnitude main shocks.

If we extend the dynamical correlations to moderate-magnitude main shocks, we need to keep the condition $\mid m \mid < M_s$, such that the distribution is

$$P_c(M_s, m) = \frac{1}{4}\beta^2 e^{-\frac{1}{2}\beta M_s} e^{-\frac{1}{2}\beta|m|} \ , \ \mid m \mid < M_s \ , \ M_s > 0 \ ; \quad (4.31)$$

purely statistical correlations are now included. The standard deviation is now $\Delta M = \Delta m = \sqrt{2}/\beta$, which leads to $\Delta M \simeq 0.61$ for the reference value $\beta = 2.3$. Such a variability of ΔM can often be found in empirical studies. For instance, from the analysis of Southern California earthquakes 1990-2001 we may infere $\beta \simeq 2$ and an average $\Delta M \simeq 0.45$ (with large errors), in comparison with the theoretical result $\Delta M = 0.7$.[15] From the New Zealand catalog (1962-1999) and Preliminary Determination of Epicentres catalog (1973-2001)[16] we may infere $\beta \simeq 2.5 - 2.3$ and an average $\Delta M = 0.43 - 0.54$, respectively, while $\Delta M = \sqrt{2}/\beta$ gives $0.56-0.61$. In other cases, like the California-Nevada data,[17] the parameters are $\beta = 2.3$ and $\Delta M \simeq 1.2$, in agreement with $\Delta M = 2\sqrt{2}/\beta$ (equation (4.30)). We note that $\Delta M = \sqrt{2}/\beta$ given here is an over-estimate, because it extends, in fact, the dynamical correlations (equation (4.25)) to small-magnitude main shocks.

[13] A. Helmstetter and D. Sornette, "Bath's law derived from the Gutenberg-Richter law and from aftershock properties", Geophsy. Res. Lett. **30** 2069 10.1029/2003GL018186 (2003).

[14] K. R. Felzer, T. W. Becker, R. E. Abercrombie, G. Ekstrom and J. R. Rice, "Triggering of the 1999 M_w 7.1 Hector Mine earthquake by aftershocks of the 1992 M_w 7.3 Landers earthquake", J. Geophys. Res. **107** 2190 10.1029/2001JB000911 (2002).

[15] A. M. Lombardi, "Probability interpretation of "Bath's law", Ann. Geophys. **45** 455 (2002).

[16] R. Console, A. M. Lombardi, M. Murru and D. Rhoades, "Bath's law and the self-similarity of earthquakes", *J.* Geophys. Res. **108** 2128 10.1029/2001JB001651 (2003).

[17] K. R. Felzer, T. W. Becker, R. E. Abercrombie, G. Ekstrom and J. R. Rice, "Triggering of the 1999 M_w 7.1 Hector Mine earthquake by aftershocks of the 1992 M_w 7.3 Landers earthquake", J. Geophys. Res. **107** 2190 10.1029/2001JB000911 (2002).

Leaving aside the dynamical correlations we are left with purely statistical correlations for clusters with moderate-magnitude main shocks. In this case we use the distribution given by equation (4.22), which leads to $\Delta M = \Delta m = 1/\sqrt{2}\beta$ and $\Delta M = 0.31$ for the reference value $\beta = 2.3$. The Bath partner for such a small value of the magnitude difference looks rather as a doublet.[18]

For statistical correlations we can compute the correlation coefficient (variance). The correlation coefficient $R = \overline{M_s M}/\Delta M_s \Delta M$ between the main shock and an accompanying event $M = M_s - \mid m \mid$ ($\mid m \mid < M_s$) computed by using the distribution given in equation (4.22) is $R = 2/\sqrt{5}$. For the correlation coefficient between two accompanying events M_1 and M_2 we need the three-events distribution (which includes $M_{1,2}$ and M_s).

According to equation (4.7), the time-magnitude distribution for small-magnitude aftershocks with changed seismicity conditions leads to a parameter b instead of the background parameter β (an increase of β). On the other hand, the dynamical correlations give $\beta/2$ instead of β (a decrease). These processes may appear with different weights. For equal weights we get a change $\frac{1}{2}(b + \beta/2) - \beta$, *i.e.* a relative change $(1-3r/2)/2r$ in the parameter β. For the reference value $r = 2/3$ this change is zero.

4.5 Foreshocks. Short-term prediction

Let us assume that we are in the proximity of a main shock with magnitude M_0, at time τ until its occurrence, and we monitor the sequence of correlated foreshocks. According to equation (4.5), the magnitudes of the (correlated) foreshocks M ($< M_0$) are related to the time τ by

$$M = \frac{1}{b}\ln(\tau/\tau_0) \ , \qquad (4.32)$$

[18]G. Poupinet, W.L. Elsworth and J. Frechet, "Monitoring velocity variations in the crust using earthquake doublets: an application to the Calaveras fault, California", J. Geophys. Res. **89** 5719 (1984); K. R. Felzer, R. E. Abercrombie and G. Ekstrom, "A common origin for aftershocks, foreshocks and multiplets", Bull. Seism. Soc. Am. **94** 88 (2004).

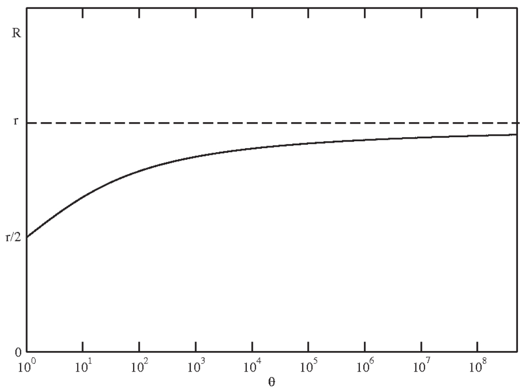


Figure 4.3: Function $R(\theta)$ vs θ for $r = 2/3$ (equation (4.35)).

where

$$\tau_0 = rt_0 e^{-b(1-r)M_0} \tag{4.33}$$

is a cutoff time, which depends on the magnitude of the main shock, the seismicity rate t_0 and the parameter $r = \beta/b$. All these parameters are provided by the analysis of the background seismic activity. The small cutoff τ_0 corresponds to a very short quiescence time before the occurrence of the main shock.[19] In addition, the time τ should be cut off by an upper threshold, corresponding to the magnitude of the main shock ($\tau < t_0 e^{\beta M_0}$). We limit ourselves to small and moderate magnitudes M in the accompanying seismic activity, such that the magnitude of the main shock may be viewed as being sufficiently large (in this respect, the so-called statistical correlations are not included).

The correlation-modified cumulative Gutenberg-Richter distribution given by equation (4.24) indicates a change in the parameter β of the standard GR distribution. We denote by B the modified parameter β; it is given by

$$e^{-\beta M} \frac{2}{1 + e^{-\beta M}} = e^{-BM} \ . \tag{4.34}$$

[19]Y. Ogata and H. Tsuruoka, "Statistical monitoring of aftershock sequences: a case study of the 2015 M_w7.8 Gorkha, Nepal, earthquake", Earth, Planets and Space **68** 44 10.1186/s40623-016-0410-8 (2016).

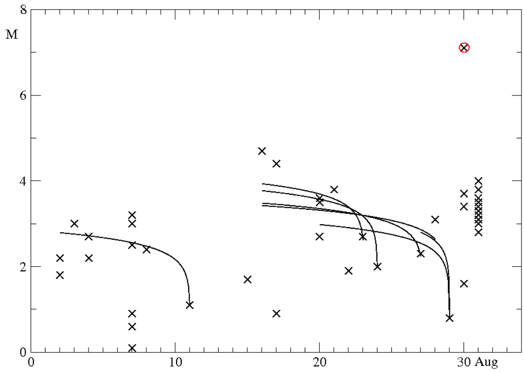


Figure 4.4: Precursory seismic events of the Vrancea 7.1-earthquake of 30 August 1986 and a series of fits by using equation (4.39).

It is convenient to introduce the ratio $R = B/b$ (similar with $r = \beta/b$ given above), such that the above equation becomes

$$R = \frac{1}{\ln\theta}\ln\left[\frac{1}{2}\left(1+\theta^r\right)\right] \ , \tag{4.35}$$

where $\theta = \tau/\tau_0$. The parameter R varies from $R = r$ for large values of the variable θ to $R = r/2$ for $\theta \longrightarrow 1$ ($\tau \longrightarrow \tau_0$). The function $R(\theta)$ is plotted in Fig. 4.3 *vs* θ for $r = 2/3$. The decrease in the function $R(\theta)$ for $\theta \longrightarrow 1$ indicates the presence of the correlations.

According to equation (4.34) the modified GR parameter B is given approximately by

$$B \simeq \beta - \frac{\ln 2}{M} \ , \tag{4.36}$$

or

$$R \simeq r - \frac{\ln 2}{bM} = r - \frac{\ln 2}{\ln(\tau/\tau_0)} \tag{4.37}$$

for a reasonable range of foreshock magnitudes $M > 1$. We can see that a 10% decrease is achieved for $M = 3$ and $\tau/\tau_0 \simeq 3.6 \times 10^4$ ($\beta = 2.3$). Such a decrease has been reported for the L'Aquila earthquake (6 April 2009, magnitude 6.3) and the Colfiorito, Umbria-Marche, earthquake (26 September 1997, magnitude 6)[20] and for the Amatrice-Norcia earthquakes (24 August 2016, magnitude 6.2; 30 October 2016, magnitude 6.6).[21] It is worth noting that smaller magnitudes occur in the sequence of correlated foreshocks for shorter times, measured from the occurrence of the main shock (the nearer main shock, the smaller correlated foreshocks).

If we update the slope B of cumulative distribution $\ln[N_{ex}(M)/N(0)] = -BM$ at various successive times t and if this B fits equation (4.36), then we may say that we are in the presence of a correlated sequence of foreshocks which may announce a main shock at the moment $t_{ms} = t + \tau$. From equation (4.32) we get

$$t_{ms} = t + \tau_0 e^{bM(t)} \ . \tag{4.38}$$

This formula provides an estimate of the occurrence moment of the main shock. It is worth noting that this moment depends on the expected magnitude of the main shock. For instance, a magnitude M indicates a time $\tau = \tau_0 e^{bM}$ up to the main shock. Let us assume that we are interested in a main shock with magnitude $M_0 = 7$; then by using $t_0 = e^{-11.32}$ (years) and $r = 2/3$ (values derived for Vrancea), we get $\tau_0 = \frac{2}{3}10^{-8.42}$ (years); a foreshock with magnitude $M = 5$ would indicate that we are at $\tau = \frac{2}{3}10^{-8.42}10^{7.5} = 0.079$ years, *i.e.* 29 days, from that main shock.[22]

Therefore, the cumulative Gutenberg-Richter distribution modified by correlations in the foreshock region and the time dependence of the foreshock magnitudes can be used to estimate the moment of

[20] A. De Santis, G. Cianchini, P. Favali, L. Beranzoli and E. Boschi, E., (2011). "The Gutenberg-Richter law and entropy of earthquakes: two case studies in Central Italy", Bull. Seism. Soc. Am. **101** 1386 (2011).

[21] L. Gulia and S. Wiemer, "Real-time discrimination of earthquake foreshocks and aftershocks", Nature **574** 193 (2019); see also L. Gulia, T. Tormann, S. Wiemer, M. Herrmann and S. Seif, "Short-term probabilistic earthquake risk assessment considering time-dependent b values", Geophys. Res. Lett. **43** 1100 (2016).

[22] B. F. Apostol and L. C. Cune, "On the time variation of the Gutenberg-Richter parameter in foreshock sequences", J. Theor. Phys. **323** (2020).

occurrence of the main shock.[23] It may be convenient to use equation

$$M = \frac{1}{b}\ln(t_{ms} - t) - \frac{1}{b}\ln\tau_0 \tag{4.39}$$

with two parameters t_{ms} and τ_0 to fit a sequence of correlated foreshocks; from $\tau_0 = rt_0e^{-b(1-r)M_0}$ (equation (4.33)) we can get the magnitude M_0 of the main shock which would occur at time t_{ms}, by using the background values of r and t_0.

From equation (4.39) we can get the time τ until a main shock with magnitude M_0 from the magnitude M measured at any time t; for $r = 2/3$ and $t_0 = e^{-11.32}$ years it is given by

$$\tau = 10^{-(2.54+M_0/2-1.5M)}\, days\ ; \tag{4.40}$$

for instance, for $M = 4$ we are at $\tau = 10^{0.04} \simeq 1$ day from a main shock with magnitude $M_0 = 7$.

Vrancea is the main seismic region of Romania. Reliable recordings of earthquakes started in Romania around 1980. Since then, three major earthquakes occurred in Vrancea: 30 August 1986, magnitude $M = 7.1$; 30 May 1990, magnitude $M = 6.9$; 31 May 1990, magnitude $M = 6.4$.[24] The 7.1-earthquake (depth $131km$) is shown in Fig. 4.4, together with all its precursory seismic events from 1 August to 31 August. All these earthquakes occurred in an area with dimensions $\simeq 100km \times 80km$ ($45^\circ - 46^\circ$ latitude, $26^\circ - 27^\circ$ longitude), at various depths in the range $30km - 170km$, except for the events of 7-8 August and the 1.6-event of 30 August, whose depth was $5km - 20km$. The subset of earthquakes from 16 August to 24 August can be fitted by equation (4.39) with the fitting parameters $t_{ms} = 24$ August, $\tau_0 = 10^{-4.76}$ days and a large rms relative error 0.32. The maximum magnitude has been used for the earthquakes which occurred in the same day, because, very likely, those with smaller magnitude are secondary accompanying events of the greatest-magnitude shock. If we assume that this is a correlated-foreshock subset, it would indicate

[23] It is worth noting that the correlation-modified magnitude distribution $P^c(M) = \beta e^{-\beta M}\frac{2}{(1+e^{-\beta M})^2}$ (equation (4.23)) is not reproduced by a standard-type Gutenberg-Richter magnitude distribution Be^{-BM} in the range $0.87 < \beta M < 2.4$.

[24] Romanian Earthquake Catalog 2018, http://www.infp.ro/data/romplus.txt.

the occurence of a main shock with magnitude 4.4 on 24 August. The main shock ($M = 7.1$) occurred on 30 August. The three earthquakes from 27 August to 29 August may belong to a subset prone to such an analysis, but it is too poor to be useful. A main shock with magnitude 7.1 ($\tau_0 = 10^{-6.06}$ days) and an average magnitude for the days with multiple events leads to a fit with a larger rms relative error 0.6. For the earthquake pair of 30-31 May 1990 (depth $87 - 91km$) we cannot identify a correlated subset of foreshocks, *i.e.* a sequence of precursory events with an average magnitude, or a maximum-magnitude envelope, decreasing monotonously in a reasonably short time range. Another particularity in this case, in comparison to the earthquake of 1986, is the quick succession (30-31 May) of two comparable earthquakes (magnitude 6.9-6.4).

In the set of precursory events of the l'Aquila earthquake, 6 April 2009 (magnitude 6.3, local magnitde 5.9) one can identify two magnitude-descending sequences, with earthquakes succeeding rapidly at intervals of hours. The first sequence occurred on 2 April, consisting of 7 earthquakes with local magnitudes from 2.1 to 1.0. The fitting of these data with equation (4.39) indicates a main shock approximately 5 hours before the earthquake with magnitude 3.0 of 3 April (with a large rms relative error 0.4). The second magnitude-descending sequence consists of 5 earthquakes with magnitudes from 1.9 to 1.1, which ocurred on 6 April. The fit, with a similar large error, indicates the occurrence of a main shock at the time $01:35$; the l'Aquila earthquake occurred at $01:32$ (UTC; the last foreshock was recorded at $01:20$). On the other hand, a magnitude-descending sequence cannot be identified before the earthquake of 4 April, with local magnitude 3.9. The data used in this analysis are taken from the Bollettino Sismico Italiano, 2002-2012, in $\pm 25km$ an area around the epicentre of the l'Aquila earthquake (42.342° latitude, 13.380° longitude). The lack of the background seismicity parameters β and $-\ln t_0$ for the l'Aquila region prevents us from estimating the magnitude of the main shocks by this analysis. We note that the magnitude in the fitting equation (4.39) is the moment magnitude; the use of local magnitudes in this equation generates (small) errors.

We applied the same procedure to the Vrancea earthquake with magnitude 3.8 (local magnitude 4.1), viewed as a main shock, which occurred on 30 November 2021.[25] By making use of the foreshock sequence from 24 November to 27 November (5 earthquakes), we can predict a main shock on 28 November, with a large magnitude (6.9, with a small rms relative error). On 28 November an earthquake with magnitude 3.1 was recorded in this area. By extending the sequence until 29 November (7 earthquakes), a main shock with magnitude 4.5 was forecasted on 1 December.[26] All these earthquakes occurred within $45° - 46°$ latitude, $26° - 27°$ longitude, at depths in the range $90km - 180km$.

The main source of errors arises from the quality of the fit $B(t)$ *vs* $M(t)$ (equation (4.36)), or, equivalently, the fit of the function $R(\theta)$ given by equation (4.35), or the fit of equation (4.39). In order to improve the quality of this fit we need a rich foreshock activity in the immediate proximity of the main shock, because the relevant part of the curve given by equation (4.39) is its abrupt decrease in the proximity of the main shock. This is an ideal situation, because the number of small-magnitude foreshocks in the immediate vicinity of the main shock is small. The fits of equation (4.39) have necessarily large errors. Another source of errors arises from the background parameters t_0 and r (β), which may affect considerably the exponentials in the formula of the time cutoff τ_0 (equation (4.33)). The procedure described above is based on the assumption that the foreshock magnitudes are ordered in time according to the law given by equation (4.32). However, according to the epidemic-type model,[27] the time-ordered magnitudes may be accompanied by smaller-magnitudes earthquakes, such that the law given by equation (4.32) may exhibit lower-side oscillations, and the slope given by equation (4.37) may exhibit upper-side oscillations.

Task #3 of Practical Seismology is a continuous monitoring of the seismic activity in a given region (say, daily, for Vrancea). At any moment we have a (necessarily short) time series of descending mag-

[25] B. F. Apostol and L. C. Cune, "Prediction of Vrancea Earthquake of November 30 2021", Seism Bull **2**, Internal Report National Institute for Earth's Physics, Magurele, (2021).

[26] All the data are taken from Romanian Earthquake Catalog 2018, http://www.infp.ro/data/romplus.txt.

[27] Y. Ogata, "Statistical models for earthquakes occurrences and residual analysis for point proceses", J. Amer. Statist. Assoc. **83** 9 (1988); "Space-time point-processes models for earthquakes occurrences", Ann. Inst. Statist. Math. **50** 379 (1998).

nitudes $M(t)$, which, introduced in equation (4.40), provides the time τ until an expected main shock with magnitude M_0. The current seismic activity in Vrancea is given in the Romanian Earthquake Catalog with a local magnitude M_L.[28] An approximate estimation of the moment magnitude M is provided by the Hanks-Kanamori law

$$M = \lg(Rv) + \frac{2}{3}\lg(8\pi\rho c^2) - 1.07 \ , \tag{4.41}$$

where R is the distance from the focus to the observation point, v is a mean amplitude of the displacement produced by the P and S seismic waves, $\rho = 5g/cm^3$ is a mean density for earth and $c = 5km/s$ is a mean velocity of the elastic waves (R and v are measured in cm, ρ is measured in g/cm^3 and c is measured in cm/s). This formula is derived in one of the next chapters. Using these values for the parameters, equation (4.41) becomes $M = \lg(Rv) - 1.8$.

4.6 Seismic activity

As a consequence of the tectonic movement, in a pointlike focus an energy E is accumulated in time t, according to the formula

$$t/t_0 = (E/E_0)^r \ . \tag{4.42}$$

The geometrical accumulation parameter r is the reciprocal of the number of dimensions of the focal region. It varies in the range $1/3 \le r \le 1$. The reference value is $r = 2/3$. The threshold energy E_0 defines the magnitude $M = 0$, according to the formula $E/E_0 = e^{bM}$, where $b = 3.45$ by convention ($b = 3/2$ for powers of ten). Introducing this formula in equation (4.42), we get the accumulation time $t = t_0 e^{\beta M}$, where $\beta = br$. This formula gives a meaning to the threshold time t_0: it shows that an earthquake with magnitude M consists of

[28] The relationship between M_L and the moment magnitude M used by the Romanian Earthquake Catalog is $M = 0.74M_L + 0.8$ for $M_L < 4.7$ and depth $h > 60km$; $M = 0.52M_L + 1.1$ for $M_L < 4.7$ and depth $h < 60km$ and $M = 1.43M_L + 2.14 - 0.018M_L^2$ for $M_L > 4.7$, any depth. These formulae should be used with caution, because they define a "moment magnitude" which depends on M_L and h.

$t/t_0 = e^{\beta M}$ seismic events with magnitude $M = 0$, each having an accumulation time t_0. The seismic activity consists of such fundamental E_0-events. In a time interval t we may have either an E_0-event with probability t_0/t, or an earthquake with energy $E > E_0$ and probability $\int_{t_0}^{t} dt'P(t')$, such that $t_0/t + \int_{t_0}^{t} dt'P(t') = 1$; hence, $P(t) = t_0/t^2$. This is the probability to have an independent earthquake at time t. Using the accumulation law we get from this probability the standard Gutenberg-Richter distributions. In particular, we get the standard cumulative (excedence) law $e^{-\beta M}$ of all the earthquakes N with magnitude greater than M. This law is fitted to empirical data, in the form $\ln(N/T) = -\ln t_0 - \beta M$, where T is the time of all the analyzed earthquakes; this way, the fitting parameters β, *i.e.* r, and t_0 are extracted. The reference value $r = 2/3$ ($\beta = 2.3$) has been established as an average value. The standard law describes the so-called background (regular) earthquakes, which are independent events.

Some earthquakes appear in the same seismic region and in a reasonably short time interval. Very likely, these earthquakes are not independent. We say that they are correlated, *i.e.* they depend on one another. It is convenient to identify the highest earthquake in magnitude in such subsets of earthquakes, which is called the main shock. The earthquakes which preced the main shock are called foreshocks, while those which succeed a main shock are called aftershocks. Regular earthquakes may also be present in the accompanying (associated) earthquakes.

The earthquake correlations may appear in various ways. If any quantity which refers to an earthquake depends on another earthquake, we say that the two earthquakes are correlated. The most general correlations are generated simply by referring an earthquake to another. This procedure is described by conditional probabilities. It is present in the time distributions of aftershocks, or foreshocks, with respect to the time τ measured from the main shock. Such a distribution, which is proportional to $1/\tau$ for small times τ (with a cutoff time), is called the Omori law. The Omori correlations are not caused by a distinct physical circumstance, but rather by the procedure we use to distribute the earthquakes. The inter-event distributions (waiting time, next earthquake distributions) are included in this class of correlations generated by conditional probabilities.

Two earthquakes may share their energy, which means a modification of the accumulation mechanism. These correlations are time-magnitude correlations. Other earthquakes may interact with one another, exchanging energy. These correlations can be seen in the statistical distributions. They are called dynamical correlations. An energy transfer is implied in both these types of correlations, although the time-magnitude correlations are deterministic, while the dynamical correlations are statistical correlations. Some other correlations appear in statistical distributions by imposing conditions on the statistical variables. We call them purely statistical correlations. They arise by the definition of the seismic events.

In general, since all these correlations are governed by physical laws which are symmetric under time reversal, we expect that the correlations are symmetric with respect to the change foreshocks-aftershocks. However, certain asymmetries appear, generated by some particularities of the two foreshock-aftershock subsets.

Let us imagine that we come from the far past, where a regular (background) regime of seismicity dominates, and approach a main shock. On approaching a main shock, higher-magnitude foreshocks appear, followed by small-magnitude foreshocks, which are very near in time to the main shock. The remaining time until the main shock is related to the magnitude of the foreshock, such that, by monitoring the foreshock activity we may estimate the occurrence moment of the main shock. This relationship between time and magnitude is governed by time-magnitude correlations, which imply a certain modification in the energy accumulation mechanism. Before any foreshock the seismicity conditions are modified, such as to accumulate a larger amount of energy. Immediately after the foreshock the seismicity conditions come back to their normal regime. This modification of the seismicity conditions is reflected in a temporary decrease in the parameter r.

At the same time, the exchange of energy between the foreshocks and the main shock generates dynamical correlations. The dynamical correlations modify the standard Gutenberg-Richter distribution, in such a way that the small-magnitude region is affected. This is called the roll-off effect in the Gutenberg-Richter distribution. The change affects mainly the region with $M \lesssim 1$. The slope β of the standard cumulative distribution remains, practically, the same. The intercept $-\ln t_0$ of the standard cumulative distribution is changed, but we have not enough data in the region $M \to 0$ to get a reliable value for this parameter. The parameter β in the small-magnitude region becomes $\beta/2$, which amounts to a reduction of r to $r/2$. This means a flattening of the standard Gutenberg-Richter distribution in the region of small magnitudes. The reduction of β for foreshocks is documented in some empirical cases. The proportion of the dynamically correlated earthquakes to the regular earthquakes can be obtained from fitting data by the corresponding distributions, each with its own weight. We expect dynamical correlations to be present in earthquake sequences accompanying high-magnitude main shocks, which, although rare, have a large productivity of small-magnitude associated earthquakes.

As regards the empirical studies it is very important to be aware of the fact that any empirical realization of the statistical ensemble of the seismic activity is a poor representation. Consequently, not all the features expected from a theory can be seen in empirical ensembles.

Very likely, the aftershocks are correlated to the main shock by time-magnitude correlations. Small-magnitude aftershocks appear immediately after the main shock, followed by higher-magnitude aftershocks. After a main shock the seismicity conditions may change, such that the cutoff time in the energy accumulation law is changed. This change leads to a modified Gutenberg-Richter distribution be^{-bM} for small-magnitude aftershocks (derived from $\left(\tau_0/\tau^2\right) d\tau$ and equation (4.32)). We can see that the number of small-magnitude aftershocks is increased, in the region of magnitudes $M < M_{2c} = \ln r/[b(r-1)]$. The range of this region is very small (for $r = 2/3$ and $b = 3/2$ we get $M_{2c} \simeq 0.36$). This increase in the aftershocks parameter β (r) is documented empirically in some cases. At the same time, the dynamical correlations may be present in aftershocks, leading to a decrease in the parameter β for small magnitudes.

The dynamical correlations may be present in aftershocks. Their lowering of the parameter β (r) for small magnitudes ($\beta \longrightarrow \beta/2$) may be compensated by the increase in this parameter, due to the change in the seismicity conditions. However, this interplay is valid over a very narrow range of small magnitudes ($\simeq 0.36$). Beyond this range we may use the reduced parameter $\beta/2$.

The dynamical correlations, which are present both in foreshocks and aftershocks, lead to a decomposition of the joint probability main shock plus accompanying earthquake, which highlights the probability distribution of the difference in magnitude between the two events. This distribution has a vanishing mean value for the magnitude difference and a deviation $\Delta M = 2\sqrt{2}/\beta$, which is the amount by which the highest aftershock (foreshock) magnitude is lower than the magnitude of the main shock. This is Bath's law. In this form it arises from dynamical correlations. However, the magnitude difference may be subject to certain conditions, either for dynamically-correlated earthquakes or for regular earthquakes. These conditions are purely statistical correlations. They lead to other values of Bath's difference, like $\sqrt{2}/\beta$, or $1/\sqrt{2}\beta$. Such a variability in Bath's difference is documented in some empirical cases. (For instance, the difference $1/\sqrt{2}\beta$ may be associated with doublets).

4.7 Watch the little ones: they may herald disasters

(contributed by M. Apostol)

A large number N of earthquakes with magnitude greater than M, occurring in a long time T in a seismic region are fitted by the well-known Gutenberg-Richter relationship

$$\lg(N/T) = -\lg t_0 - \beta M \quad , \tag{4.43}$$

where t_0 and β are fitting parameters. For instance, a set of 3640 earthquakes with magnitude $M \geq 3$, which occurred in Vrancea between 1981 and 2018, leads to $-\lg t_0 = 4.92$ (t_0 measured in years) and $\beta = 0.98$, with an estimated 15% error; a useful parameter is $r = 2\beta/3 \simeq \frac{2}{3}$.

The earthquakes may be correlated: a main shock may be preceded by correlated, smaller foreshocks, and it may be followed by correlated, smaller aftershocks. According to the theory, the time τ elapsed from the occurrence of a foreshock with magnitude M to the occurrence of a main shock with magnitude M_0 (or the time from an M_0-main shock to an M-aftershock) is given by

$$\tau = rt_0 \cdot 10^{-\frac{3}{2}[(1-r)M_0 - M]} \ , \qquad (4.44)$$

or, with the parameters given above,

$$\tau = 10^{-(2.54 + M_0/2 - 3M/2)} \, days \ . \qquad (4.45)$$

The background parameters t_0 and r (β) can be derived for any seismic region (the parameter r varies in the range $\frac{1}{3} < r < 1$). By inserting them in equation (4.44), we can estimate the time which may elapse from an M-earthquake to an M_0-main shock (if they are correlated). We can see that in any moment we may expect a large earthquake. The time τ is shorter for a greater M_0 and a smaller M. The smaller the correlated foreshock, the nearer a large main shock: watch the little ones, they may announce disasters. If $M \longrightarrow 0$ is correlated with an M_0-main shock, we are already at the moment $\tau_0 = 10^{-(2.54+M_0/2)}$ days when this main shock occurrs (a very short time). We do not know the moment of occurrence of the earthquakes. At any moment a "big one" may appear.

For instance, an $M = 3$-foreshock may announce an $M_0 = 7$ main shock after 41.5 minutes; or an $M_0 = 6$-main shock after 2.19 hours; or an $M_0 = 5$-main shock after $\simeq 7$ hours.

5 Entropy of Earthquakes

5.1 Thermodynamic ensembles

Let us consider a gas of $N \gg 1$ identical, non-interacting particles, moving in a large volume V (in the three-dimensional space), each within a large range of momenta $\boldsymbol{p}$. We consider the coordinates $\boldsymbol{q}$ and the momenta $\boldsymbol{p}$ as undetermined statistical variables. We say that each particle is a statistical sub-ensemble and the gas is a statistical ensemble. This is the starting framework of Statistical Physics, as formulated by Gibbs.[1]

As any statistical variable, the coordinates $\boldsymbol{q}$ and the momenta $\boldsymbol{p}$ of each particle should be distributed by a probability density ρ, which should be normalized. The normalization condition is

$$\int \rho \cdot d\boldsymbol{p}d\boldsymbol{q}/(2\pi\hbar)^3 = 1 \; ; \tag{5.1}$$

the Planck constant $\hbar$ measures the unit cell in the phase space (of generalized coordinates $\boldsymbol{p}$ and $\boldsymbol{q}$), according to the uncertainty principle of Quantum Mechanics; it makes the probability density dimensionless, as it should be. The point $(\boldsymbol{p}, \boldsymbol{q})$ defines a state in the phase space (for a classical particle). Each particle is free to acquire any state in the whole volume and the whole range of momenta.

If a particle is free, it is conceivable that it moves perfectly chaotical, *i.e.* it is subject to a maximal disorder; if it has at its disposition a number Γ of states, then its probability density should be $\rho = 1/\Gamma$. We note that this probability density, of perfectly randomly distributed events, should be minimal. We have $d\Gamma = d\boldsymbol{p}dq/(2\pi\hbar)^3$. The hypothesis of a perfect chaos (called "molecular chaos") has been introduced

[1] J. W. Gibbs, *Elementary Principles in Statistical Mechanics*, Scribner's sons, NY (1902).

in Statistical Physics by Boltzmann.[2] The use of the number of states Γ opened the way towards Quantum Statistics.

Let us assume that we put together two such particles, denoted by 1 and 2. If they are free, we can define for each a number of available states $\Gamma_{1,2}$, the same as if they were separated; their probability densities are $\rho_{1,2} = 1/\Gamma_{1,2}$, since, being free, they move perfectly chaotical. In addition, being free, they are independent, such that the number of available states of the ensemble of the two particles is $\Gamma = \Gamma_1\Gamma_2$ and the probability density of the ensemble is $\rho = 1/\Gamma_1\Gamma_2 = \rho_1\rho_2$. It is convenient to use $-\ln\rho$ $(= \ln\Gamma)$ instead of ρ, because $\ln\rho$ is additive for multiplicative ρ's. $S = -\ln\rho$ is called entropy. We see that it should be maximal, according to the hypothesis of the perfect chaos. This is the law of increase of entropy. Also, we see that this hypothesis implies free, non-interacting and independent particles. We note that this hypothesis does not exclude certain conditions in defining the states of the particles, the same for each particle. (Such conditions are encountered in Quantum Statistics, and are called sometimes quantum-mechanical correlations).

The hypothesis described above is called statistical-independence hypothesis. It defines the entropy as an additive quantity (like other physical quantities, *e.g.* the energy). The corresponding ensemble is said to be extensive. Extensive statistical ensembles are called thermodynamic ensembles.

It is worth noting that the assumption of a given number of available states Γ defines an external condition needed to characterize the ensemble. We need external conditions to define ensembles. For instance, we may use a fixed energy as an external condition. Then, we have a micro-canonical ensemble. The entropy defined above is often called a micro-canonical entropy. We may imagine an ensemble in contact with a so-called "thermal bath", exchanging continuously energy with that bath, *i.e.* giving and receiving energy; preserving at the same time the hypothesis of a perfect chaos. Such an ensemble is called canonical ensemble (if it exchanges also particles, it is called grand-canonical ensemble). It is worth noting that this hypothesis for

[2] L. Boltzmann, *Lectures on Gas Theory*, Dover, NY (1964) (translated from L. Boltzmann, *Vorlesungen uber Gastheorie*, Barth, Leipzig, Part I (1896) and Part II (1898)).

a canonical ensemble makes it stationary in time (because a perfect chaos is unique); we call this condition statistical, or thermodynamic equilibrium. The entropy in this case should also be maximal, only that we need a mean entropy, defined as $-\int d\Gamma\rho\ln\rho$. We note that the entropy defined above as $S=-\ln(1/\Gamma)$ is a mean entropy, because $-\int d\Gamma\rho\ln\rho=-\Gamma\cdot\frac{1}{\Gamma}\ln(1/\Gamma)=-\ln(1/\Gamma)$. It follows that we can compute the entropy of a canonical ensemble by imposing upon it a stationarity condition under the constraints of a fixed mean energy and the normalization (in fact, this condition is a maximum condition with respect to the probability density, as for a maximal disorder). The stationarity of the entropy in equilibrium has been introduced in Thermodynamics by Clausius.[3] Therefore, let us seek the stationarity of the Lagrange function

$$S=-\int d\Gamma\rho\ln\rho-\alpha\left(\int d\Gamma\rho-1\right)-\beta\left(\int d\Gamma\rho\mathcal{E}-E\right)\ , \qquad (5.2)$$

where $\mathcal{E}$ is the statistical variable of the energy, E is the mean energy and α, β are Lagrange's coefficients to be determined. We get

$$\rho=C(\beta,V)e^{-\beta\mathcal{E}} \qquad (5.3)$$

and

$$S=\beta E-\ln C\ , \qquad (5.4)$$

where C (coefficient α) is determined from $\int d\Gamma Ce^{-\beta\mathcal{E}}=1$ (and the mean energy E is a function of β and V); the coefficient β is determined from the mean energy (which is an external condition, like V). We note that these equations are valid both for statistical subensembles and statistical ensembles. Equation (5.3) gives the canonical Boltzmann-Gibbs distribution. (The proportionality $\ln\rho\sim-E$ follows also from the stationarity of the two conserved quantities ρ and E, since, for a mechanical ensemble, $\ln\rho$ should be a linear combination of the prime integrals). It is convenient to write $C=e^{\beta F}$ and introduce the temperature $T=1/\beta$; then, we have $F=E-TS$

[3]R. Clausius, "Uber verschiedene fur die Anwendung bequeme Formen der Hauptgleichungen der mechanischen Warmetheorie", Ann. Phys. **125** 353 (1865); *The Mechanical Theory of Heat - with its Applications to the Steam Engine and to Physical Properties of Bodies*, van Voorst, London (1867).

and

$$dE = \frac{\partial F}{\partial V}dV + TdS = -pdV + TdS \tag{5.5}$$

at fixed T, where p, by definition, is the pressure. F is called the free energy. This is the thermodynamic equation, which defines an equilibrium transformation. TdS is the heat. We can see how the Statistical Physics produces Thermodynamics.

In equilibrium $dS = 0$. The variation of the entropy with respect to the equilibrium is

$$\delta^{(2)}S = -\int d\Gamma(1/\rho)(\delta\rho)^2 < 0 \ , \tag{5.6}$$

which shows that the entropy is maximal in equilibrium (with respect to the variation of the probability density).

In an equilibrium transformation

$$dS = \frac{1}{T}(dE + pdV) \ . \tag{5.7}$$

If $dS = 0$ the transformation is reversible. If the changes are very slow, the transformation is adiabatic (and the process is a quasi-equilibrium process).

If the ensemble deviates from equilibrium, the entropy should decrease by an amount δS given by

$$\Delta S = \frac{1}{T}(\Delta E + p\Delta V) - \delta S \ . \tag{5.8}$$

It follows, that the probability density of the deviations of the mean quantities is proportional to

$$e^{-\delta S} = e^{-\frac{1}{T}(\Delta E + p\Delta V - T\Delta S)} \ ; \tag{5.9}$$

the exponent in this equation is proportional to

$$\Delta E + p\Delta V - T\Delta S = \frac{1}{2}\left(\frac{\partial^2 E}{\partial V^2}\Delta V^2 + 2\frac{\partial^2 E}{\partial V \partial S} + \frac{\partial^2 E}{\partial S^2}\right) \ . \tag{5.10}$$

This equation leads to the normal distribution of the fluctuations (known as Einstein's fluctuation formula[4]). Being additive, this variation is proportional to N. Therefore, the dispersion of the fluctuations

[4] A. Einstein, "Zum gegenwaertigen Stand des Strahlungsproblem", Phys. Z. **10** 185 (1909).

is proportional to $1/\sqrt{N}$; in the limit of large N the thermodynamic fluctuations are vanishing. This is why we need a large N. (For small values of N the mesoscopic, or nanoscopic ensembles exhibit large fluctuations).

5.2 Seismic activity

Following the instance offered by a gas of particles many other statistical ensembles can be constructed. Some of them are not thermodynamic (extensive) ensembles. This is the case of the seismic activity. Earthquakes occur at undetermined moments of time with undetermined magnitudes. This induces the idea that they may be described by a probability density with the magnitude as the statistical variable (occurrence time, energy, location, etc can also be viewed as statistical variables for earthquakes). Moreover, the earthquakes are produced by the movement of the tectonic plates, so they may be viewed as a canonical ensemble. The tectonic bath feeds continuously the earthquake activity, but the energy released by earthquakes does not go back to the tectonic plates, at least not integrally, nor directly. This special circumstance has important consequences.

In order to be able to define a probability we need empirical realizations of the statistical ensemble of the seismic activity as large as possible, with many earthquakes for each magnitude value. Such a realization can be obtained by collecting all the earthquakes which occurred in a given seismic region in a given, long period of time. We assume that the magnitudes are measurable with a reasonable accuracy. We need also reproducibe results of the measurements, so we need many realizations of the same statistical esemble in the same given conditions. This is not possible with the seismic activity, because the seismicity conditions change in time. However, many updates of the statistical ensemble of a given seismic region may be viewed as many realizations of the same statistical ensemble.

The seismic activity is one statistical ensemble which produces various values of the magnitude. It is not formed by several statistical subensembles; it is only one ensemble. Therefore, the seismic acivity may be a statistical ensemble, but it is not a thermodynamic (extensive) ensemble.

Let us assume that the seismic activity is a canonical ensemble in equilibrium. Consequently, it is described by an entropy S, which, under the constraint of a fixed mean magnitude $\overline{M}$, leads, by its maximization, to[5]

$$S = \beta\overline{M} - \ln\beta \ , \tag{5.11}$$

exactly as in equations (5.2) to (5.4), where C is equal to β. The resulting probability density is

$$\rho = \beta e^{-\beta M} \ . \tag{5.12}$$

This is precisely the Gutenberg-Richter magnitude distribution, which is well-documented empirically.[6] From empirical studies the value $\beta = 2.3$ is accepted as a reference value. The corresponding cumulative Gutenberg-Richter distribution is $e^{-\beta M}$, given empirically by

$$\ln N(M) = \ln N(0) - \beta M \ , \tag{5.13}$$

where $N(M)$ is the number of earthquakes with magnitude greater than, or equal to M, out of a total number of earthquakes $N(0)$.

From equation (5.12) the mean magnitude is $\overline{M} = 1/\beta$ (and $S = 1 - \ln\beta$ in equilibrium). In equilibrium $(\partial S/\partial\beta)_{\overline{M}} = \overline{M} - 1/\beta = 0$. The variation of the entropy with respect to the equilibrium is

$$\Delta S = (\overline{M} - 1/\beta)\Delta\beta + \frac{1}{2\beta^2}(\Delta\beta)^2 + ... = \frac{1}{2\beta^2}(\Delta\beta)^2 + ... \ ; \tag{5.14}$$

it follows the fluctuation distribution

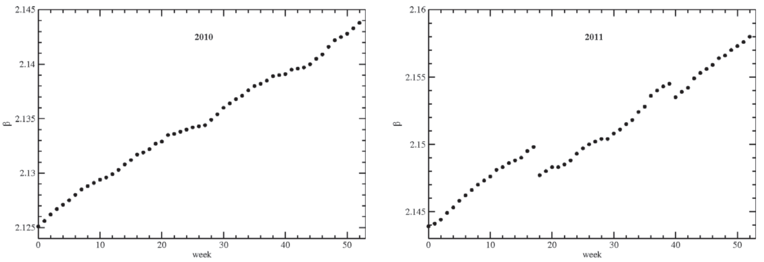
$$\rho_f = \frac{1}{\sqrt{2\pi}\beta} e^{-\frac{(\Delta\beta)^2}{2\beta^2}} \ . \tag{5.15}$$

The standard deviation of the parameter β is

$$\delta\beta = \left[\overline{(\Delta\beta)^2}\right]^{1/2} = \beta \ ; \tag{5.16}$$

[5] B. F. Apostol and L. C. Cune, "Entropy of earthquakes: application to Vrancea earthquakes", Acta Geophys. doi: 10.1007/s11600-021-00550-4 (2021).

[6] B. Gutenberg and C. Richter, "Frequency of earthquakes in California", Bull. Seism. Soc. Am. **34** 185 (1944); "Magnitude and energy of earthquakes", Ann. Geofis. **9** 1 (1956) (Ann. Geophys. **53** 7 (2010)); C. F. Richter, *Elementary Seismology*. Freeman, San Francisco, California (1958); G. Ranalli, G., (1969). "A statistical study of aftershock sequences", Ann. Geofis. **22** 359 (1969).



(a) (b)

Figure 5.1: Short-time (weeks) continuous (panel a, year 2010) and discontinuous (ruptures, panel b, year 2011) variations of the parameter β for Vrancea (Romanian Earthquake Catalog, 2018, updated).

its finite value shows that the seismic activity is a non-extensive statistical ensemble.

If the entropy is viewed as measuring the degree of disorder, then $S = 1 - \ln\beta > 0$, *i.e.* $\beta < e$ (this condition corresponds to the classical-gas condition in Statistical Physics). For statistical ensembles the Shannon entropy $\Sigma = -\sum_n p_n \ln p_n$ is often used, where $p_n = \left(1 - e^{-\beta\delta}\right) e^{-n\beta\delta}$ is the discrete Gutenberg-Richter magnitude probability with step δ (properly normalized for $n = 0, 1, 2...$).[7] This entropy depends on the arbitrary discretization step δ, $\Sigma = S - \ln\delta + ...$ (for $\beta\delta \ll 1$), where $S = 1 - \ln\beta$. The discretization step δ corresponds to Planck's constant in Quantum Statistics.

[7] A. De Santis, C. Abbattista, L. Alfonsi, L. Amoruso, S. A. Campuzano, M. Carbone, C. Cesaroni, G. Cianchini, G. De Franceschi, A. De Santis, R. Di Giovambattista, D. Marchetti, L. Martino, L. Perrone, A. Piscini, M. L. Rainone, M. Soldani, L. Spogli and F. Santoro, "Geosystemics view of earthquakes", Entropy **21** 412 (2019) doi:10.3390/e21040412.

5.3 Vrancea non-equilibrium seismic activity

Let us assume that we fitted the Gutenberg-Richter distribution (*e.g.*, the cumulative distribution, equation (5.13)) to data gathered over a long period of time t_0 for a given region. At the moment of time t_0 we have the fitting parameter β_0. For a sufficiently long period of time t_0 we may assume that this seismic activity is statistically well-defined. Let us take (at random) the next moments of time t_i, $i = 0, 1, 2, ...N$ and update the fitting to get the parameters β_i. If the time intervals $t_{i+1} - t_i$ are sufficiently small (but still as large as to have a measurable seismic activity in each) we may expect that the variations $\Delta\beta_i = \beta_{i+1} - \beta_i$ are fluctuations. For a sufficiently large N we may fit their distribution with the normal law given by equation (5.15). Thus, we get the fitting parameter β. If $\beta = \beta_0$ (within the fitting errors), the seismic equilibrium has not changed. If $\beta \neq \beta_0$ the equilibrium has changed over the period $t_N - t_0$. Consequently (in the absence of an external agent), we may expect a tendency to recover the equilibrium over the next period of time $t_{2N+1} - t_{N+1}$. Such an information might be regarded as a short-time prediction. For instance, if $\beta < \beta_0$, the mean magnitude increased, so we may expect in the next time interval a decrease in the mean magnitude, *i.e.* the number of earthquakes with low magnitudes will increase, and high-magnitude earthquakes are not likely. On the contrary, if $\beta > \beta_0$, then the mean magnitude decreased, and we may expect an increase in the number of earthquakes with higher magnitude. We note that an increase (decrease) in β amounts to a decrease (increase) in equilibrium entropy $S = 1 - \ln\beta$ (equation (5.11)).

It may happen that the distribution of the parameter changes $\Delta\beta_i$ is not a normal distribution. Then, the ensemble is not in equilibrium in the time period $t_N - t_0$. Under the equilibrium hypothesis, we may expect an evolution towards equilibrium in the next period. For instance, if the distribution of the variations $\Delta\beta_i$ is shifted towards higher values, *i.e.*, if the parameters β_i show a tendency to increase, we may expect a decrease of these parameters in the next period, *i.e.* an increase in the mean magnitude.

It is worth noting that the discussion given above is valid under the main asumption of independent seismic events. If correlations exist, the entropy formulae derived above do not apply. A special case in this connection is the short-term foreshock (and aftershock) activity. The accompanying seismic activity obeys, approximately, the Gutenberg-Richter magnitude distribution,[8] and a decrease in the parameter β, observed for the foreshock activity, was interpreted as an increase in entropy.[9] Moreover, recently it was shown that a real-time discrimination between foreshocks and aftershocks might be attained by monitoring the variations in the parameter β .[10]

If the correlations are included, we expect a change in the distribution. In this case, the formulae given above for the entropy are not valid anymore. It was shown that the change caused by correlations in the Gutenberg-Richter distribution affects mainly the small-magnitude region.[11] For moderate and large earthquakes the distribution preserves its independent-event form $\sim e^{-\beta M}$, which ensures the validity of the entropy formulae used here. Small-magnitude earthquakes ($M < 2$) are excluded from our analysis.

Also, we note that the practical application of the procedure described above depends on the choice of the (long) time period t_0, the (short) time intervals $t_{i+1} - t_i$ and the (large) number N of these intervals. This choice can only be made in close connection with the particular character of the seismic activity in the given region and in the given (long) time period.

The description given above is not supported by data, at least for Vrancea region, in the analyzed time periods. The parameter β is quasi-uniformly increasing in time, at a slow rate, due to the accumulation of small-magnitude earthquakes. This quasi-uniform tendency is interrupted from time to time by large-magnitude earthquakes, which decrease suddenly the parameter β. We may say that

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[8] C. Kisslinger, "Aftershocks and fault-zone properties", Adv. Geophys. **38** 1 (1996).

[9] A. De Santis, G. Cianchini, P. Favali, L. Beranzoli and E. Boschi, "The Gutenberg-Richter law and entropy of earthquakes: two case studies in Central Italy", Bull. Sesim. Soc. Am. **101** 1386 (2011).

[10] L. Gulia and S. Wiemer, "Real-time discrimination of earthquake foreshocks and aftershocks", Nature **574** 193 (2019).

[11] B. F. Apostol, "Correlations and Bath's law", Res. Geophys. Sci. **5** 100011, (2021).

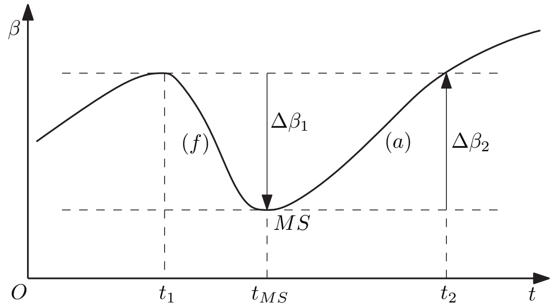

Figure 5.2: A qualitative sketch of the variation of the parameter β *vs* time t around the moment t_{MS} of the occurrence of a main shock (MS), possibly including a foreshock region (f) and an aftershock region (a). The total variation of the parameter β in the time intervals (t_1, t_{MS}) and (t_{MS}, t_2) is zero ($\Delta\beta_1 + \Delta\beta_2 = 0$), indicating an equilibrium process (constant entropy, $\Delta S = 0$).

the seismic activity in Vrancea is not in equilibrium; it is at most in quasi-equilibrium, as the time variations of the parameter β are very slow. Two examples of short-time variations of beta are given in Fig. 5.1 for Vrancea seismic activity. The time t_0 is from 1 January 1980 to 31 December 2009, with 5391 earthquakes with magnitude greater than 2. The Gutenberg-Richter fit to these data gives $\beta_0 = 2.121$ (error 15%). We have updated the parameter β for each week of the year 2010 (Fig. 5.1, panel a). We can see that this parameter increases continuously over this whole year (due to the accumulation of small-magnitude earthquakes). A similar procedure was used for each of the next years up to 2019 (8455 earthquakes with $M \geq 2$ in the whole period $1980 - 2019$, taken from Romanian Earthquake Catalog, 2018, updated. In some years the continuous increase of the parameter β is disrupted by the occurrence of greater-magnitude earthquakes, like in the year 2011 (Fig. 5.1, panel b). Such variations of the parameter β cannot be fitted by a normal distributions, and, therefore, they cannot be viewed as fluctuations. We can only say, very imprecisely and qualitatively, that after a period of small-magnitude seismic activity it is likely to follow a few earthquakes with greater magnitude, and viceversa, which is a useless, common-sense expectation. Rigorously speaking, the seismic activity is not in (quasi-) equilibrium, because the tectonic energy source feeds it continuously. We expect this behaviour to have a general character. We note that such abrupt variations in the parameter β (and an abrupt increase in the entropy) have been reported for the accompanying seismic activity of the L'Aquilla earthquake (magnitude 6.3, 6 April 2009) and the Colfiorito earthquake (magnitude 6, 26 September 1997).[12]

The overall variation of the parameter β is a slow increase in time, which may suggest a quasi-equilibrium adiabatic process. In the neighbourhood of a greater earthquake the parameter β suffers an abrupt variation. A qualitative sketch of the typical variations of the parameter β for a time interval (t_1, t_2) which includes a main shock (MS) at the moment t_{MS} is given in Fig. 5.2. After a (slight) increase the parameter β suffers an abrupt decrease $\Delta\beta_1$, possibly in a foreshock region, down to the main shock, followed by an abrupt increase $\Delta\beta_2$ which may include, possibly, an aftershock region. We can see that the total variation $\Delta\beta = \Delta\beta_1 + \Delta\beta_2 = 0$, such that we may say that over this region there exists a (quasi-) equlibrium process ($\Delta S = 0$). It was suggested to use the precursory decrease in the foreshock region, distinct from an increase in the aftershock region, as a real-time prediction of a main shock.[13]

There exists another interpretation of the sudden variation of the parmeter β in the vicinity of a large-magnitude earthquake, where the continuous variable is the time t (Fig. 5.1, panel b). Indeed, the sudden jump in the parameter β indicates a time derivative $\frac{\partial\beta}{\partial t} = \Delta\beta_1\delta(t - t_{MS})$, with the notations in Fig. 5.2. This abrupt variation induces a similar variation in the time derivative of the entropy $\frac{\partial S}{\partial t} = -\frac{\Delta\beta_1}{\beta_1}\delta(t - t_{MS})$. The singularity indicated by the function $\delta(t - t_{MS})$ is associated with a phase-transition singularity, which would correspond to a critical regime.[14] However, this is not a ther-

[12]A. De Santis, G. Cianchini, P. Favali, L. Beranzoli and E. Boschi, "The Gutenberg-Richter law and entropy of earthquakes: two case studies in Central Italy", Bull. Sesim. Soc. Am. **101** 1386 (2011).

[13]L. Gulia and S. Wiemer, "Real-time discrimination of earthquake foreshocks and aftershocks", Nature **574** 193 (2019).

[14]A. De Santis, C. Abbattista, L. Alfonsi, L. Amoruso, S. A. Campuzano, M. Carbone, C. Cesaroni, G. Cianchini, G. De Franceschi, A. De Santis, R. Di

modynamic interpretation of the singularity, because the continuous parameter is the time, not the temperature $T = 1/\beta$, which has also a jump (it is discontinuous) at t_{MS}.

The absence of fluctuations in the seismic activity analyzed here raises an interesting question. In Statistical Physics (Thermodynamics) the fluctuations are analyzed for an ensemble consisting of a large number $N \gg 1$ of sub-ensembles (sub-systems). The standard deviation $\delta\beta = \beta$ computed in equation (5.16) corresponds to one sub-ensemble. The sub-ensemble method is convenient for an extensive ensemble (like a gas of N particles). The seismic activity lacks this extensive property. However, we may view the successive updates of the parameter β described above as a series of N distinct, random, independent realizations of our ensemble, such that the average value of the parameter β is given by

$$B = \frac{1}{N}\sum_{i=1}^{N}\beta_i \ , \tag{5.17}$$

with the mean value $\overline{B} = \overline{\beta}$. We note that these assumptions imply the statistical equilibrium. The mean squre deviation of B is

$$\overline{\Delta B^2} = \frac{1}{N^2}\sum_{i,j=1}^{N}\overline{\Delta\beta_i\Delta\beta_j} = \frac{1}{N^2}\sum_{i=1}^{N}\overline{\Delta\beta_i^2} \ , \tag{5.18}$$

which can be written as $\overline{\Delta B^2} = \frac{1}{N}\overline{\Delta\beta^2}$; hence, we find $\delta B/\overline{B} = \sqrt{\overline{\Delta B^2}}/\overline{B} = \frac{1}{\sqrt{N}}\left(\delta\beta/\overline{\beta}\right)$, where we may take $\delta\beta = \beta$ and $\overline{\beta} = \beta$. We can see that the relative fluctuation of B is vanishing for large N and the dispersion of the variables β_i is $\delta\beta = \sqrt{\overline{\Delta\beta^2}} = \beta$. This corresponds to the normal (gaussian) distribution given by equation (5.15) (this result is also known as the central limit theorem). If the normal distribution changes in time, under the assumption of (quasi-) statistical equilibrium, we would have the possibility to do a prediction, as discussed above. From the discussion given above we see that this picture is not supported by data. The values β_i of the

Giovambattista, D. Marchetti, L. Martino, L. Perrone, A. Piscini, M. L. Rainone, M. Soldani, L. Spogli and F. Santoro, "Geosystemics view of earthquakes", Entropy **21** 412 (2019) doi:10.3390/e21040412.

parameter β exhibit a slight, uniform increase in time, interrupted by disparate, aparently random abrupt decreases. This behaviour indicates a non-equilibrium process.

The time variation of the parameter β of the Gutenberg-Richter distribution may be used as a quantitative measure of the departure from equilibrium of a given seismic activity. For example, from Fig. 5.1 we can estimate a variation $\simeq 1\%$ for β during the year 2010 and $\simeq 0.5\%$ for year 2011. The latter is smaller, due to the occurence of two large earthquakes in 2011, which lowered the parameter β (1 May 2011, magnitude 4.9 and 4 October 2011, magnitude 4.8). Similar values are obtained for other years, which may indicate that the seismic activity in Vrancea has a rather steady character, with a constant rate of change in time of the magnitude distribution, over a long period. Although these figures are very small, and we might be tempted to assign a quasi-equilibrium character to the seismic activity, such a conclusion is not supported by the lack of fluctuations; the existence of the fluctuations is a necessary element for the statistical (quasi-) equilibrium. However, if we view the small-earthquake increase in the parameter β together with the decrease brought about by larger earthquakes as long-period quasi-oscillations over a long period of time (including a large set of data), then we may assume that these quasi-oscillations are fluctuations. Such quasi-oscillations have been identified previously on the data corresponding to 3640 earthquakes with magnitude $M \geq 3$ which occurred in Vrancea during $1981 - 2018$.[15] It seems that such quasi-oscillations have also been seen, although on small-sized samples.[16] From this perspective, long-period quasi-oscillations in the parameter β, corroborated with a small, overall increase, might lead to assuming that the seismic activity may be approximated by a quasi-equilibrium process over such very long periods of time. However, we note that the steady increase of the parameter β and its disparate ruptures are not easily distributed on a normal gaussian.

The maximization of the canonical entropy under the condition of a fixed mean energy (magnitude) leads to the canonical distribution, as described above. This distribution fits the emprical data, except

[15]B. F. Apostol, *Seismology*, Nova, NY (2020).

[16]I. Main and F. Al-Kindy, "Entropy, energy and proximity to criticality in global earthquake populations", Geophys. Res. Lett. **29** (7) 10.1029/2001GL014078 (2002).

for small-magnitude region, where there exist correlations (the roll-off effect). This indicates that the seismic events are not independent, especially for small magnitudes, which are the most numerous. Consequently, the seismic activity is not in equilibrium.

Task #4 of Practical Seismology follows from the continuous updating (daily) of the parameter β for Vrancea. The weekly updatings should be plotted, in order to get the annual variation of the parameter β, which gives information about the state of the non-equilibrium process of the seismic activity. For instance, we can estimate the annual increase of the parameter β, as described above.

6 Seismic Moment. Seismic Waves

6.1 Focal force

There exist earthquakes which are produced by forces localized, for a short time, in a focal region with small dimensions, called the earthquake focus. The dimensions of the focus are small in comparison with distances where we measure the effects of the earthquake on Earth's surface. Similarly, the duration of the earthquake is short in comparison with the relevant times during which we measure the effects of the earthquake. We call these earthquakes tectonic, elementary earthquakes. There exist other classes of earthquakes, whose focal regions are distributed over larger regions (like, for instance, some surface earthquakes), or with the focal region propagating along large ruptures.

The position of the focus and the duration of the effects of the earthquakes on Earth's surface are determined by seismographs, which record seismograms. A typical seismogram exhibits a weak, preliminary tremor, consisting of longitudinal P ("primary") waves and transverse S ("secondary") waves, followed by a large main shock,[1] as shown in Fig. 6.1. (This "main shock" is different from the "main shock" used for foreshock-main shock-aftershock sequences). At any point of a region with reasonable dimensions about the epicentre these seismic waves come and go, pass over these points. They

[1] R. D. Oldham, *Report on the Great Earthquake of 12th June, 1897*, Geol. Surv. India Memoir 29 (1899); "On the propagation of earthquake motion to long distances", Trans. Phil. Roy. Soc. London **A194** 135 (1900); C. G. Knott, *The Physics of Earthquake Phenomena*, Clarendon Press, Oxford (1908); A. E. H. Love, *Some Problems of Geodynamics*, Cambridge University Press, London (1926).

disappear gradually by exciting the eigenmodes of vibrations of the (quasi-) spherical Earth, which may last a long time (raising, finally, the Earth's temperature). The seismic waves and the main shock are propagating waves. The effects of the earthquakes on Earth's surface are analyzed often in terms of Rayleigh surface "waves",[2] which are vibrations. Herein, we limit ourselves to the transient regime of propagation of the elastic waves, prior to the establishment of the regime of stationary vibrations.[3]

In regard to the propagation of the seismic waves the Earth may be viewed as a homogeneous and isotropic elastic body, with a mean density $\rho = 5g/cm^3$. The velocity of the longitudinal waves is $c_l \simeq 7km/s$ and the velocity of the transverse waves is $c_t \simeq 3km/s$. Also, for distances of reasonable magnitudes, the Earth may be viewed as a semi-infinite space (a half-space).

The main problem of Seismology is to account for the recordings of the seismographs, *i.e.* for the P and S seismic waves and the main shock. This is known as the Lamb problem (or seismological problem).[4] In 1848 Kelvin computed the deformation of an infinite (homogeneous, isotropic) elastic body for a point force,[5] and Stokes derived in 1849

[2]Lord Rayleigh, "On waves propagated along the plane surface of an elastic solid*"*, Proc. London Math. Soc. **17** 4 (1885) (J. W. Strutt, Baron Rayleigh, *Scientific Papers*, vol. 2, Cambridge University Press, London (1900), p. 441.

[3]B. F. Apostol, "On the Lamb problem: forced vibrations in a homogeneous and isotropic elastic half-space", Arch. Appl. Mech. **90** 2335 (2020).

[4]R. D. Oldham, "On the propagation of earthquake motion to long distances*"*, *Trans. Phil. Roy. Soc. London* **A194** 135 (1900); H. Lamb, "On wave-propagation in two dimensions*"*, Proc. Math. Soc. London **35** 141 (1902); "On the propagation of tremors over the surface of an elastic solid*"*, Phil. Trans. Roy. Soc. (London) **A203** 1 (1904); A. E. H. Love, "The propagation of wave-motion in an isotropic elastic solid medium", Proc. London Math. Soc. (ser. 2) **1** 281 (1903); H. Jeffreys, "On the cause of oscillatory movement in seismograms*", Monthly Notices of the Royal Astron. Soc., Geophys. Suppl.* **2** 407 (1931). See also M. Bath, *Mathematical Aspects of Seismology*, Elsevier, Amsterdam (1968); A. Ben-Menahem and J. D. Singh, *Seismic Waves and Sources*, Springer, NY (1981); K. Aki and P. G. Richards, *Quantitative Seismology*, University Science Books, Sausalito, CA (2009).

[5]Sir William Thompson, Lord Kelvin, "On the equations of equilibrium of an elastic solid", Cambridge Dublin Mathematical Journal **3** 87 (1848); *Mathematical and Physical Papers*, vol. 1, Cambridge Univ. Press, London (1982), p. 97.

the spherical elastic waves generated in such a body by a point force.[6] However, a force is not allowed, because it would give a translation motion of the Earth as a whole; also, a non-vanishing torque of forces is forbidden, because it would give a rotation of the Earth as a whole. Let us assume a force density $\boldsymbol{f}(t)w(\mathbf{R})$ at position $\boldsymbol{R}$ and time t, where $\boldsymbol{f}$ is the force and w is a distribution function. The net force acting along the i-th direction on an infinitesimal volume can be represented as

$$f_i w(x_1+h_1, x_2+h_2, x_3+h_3) - f_i w(x_1, x_2, x_3) \simeq \\ \simeq f_i h_j \partial_j w(x_1, x_2, x_3) \ , \tag{6.1}$$

where f_i, $i=1,2,3$, are the components of the force, h_j, $j=1,2,3$, are the components of an infinitesimal displacement $\mathbf{h}$ and x_i are the coordinates of the point with the position vector $\mathbf{R}$. It is convenient to generalize the product $f_i h_j$ to a tensor M_{ij}, which is assumed to be a symmetric tensor. In addition, the distribution function w is taken as the Dirac delta function $w(\boldsymbol{R}) = \delta(\boldsymbol{R} - \boldsymbol{R}_0)$, where $\boldsymbol{R}_0$ is the position of the focus. Therefore, the force density in the focus can be written as

$$F_i(\mathbf{R}, t) \quad = \quad M_{ij}(t)\partial_j \delta(\mathbf{R} - \mathbf{R}_0) \ , \tag{6.2}$$

where $M_{ij}(t)$ includes the time dependence. Further on, we may assume that this force acts a very short time T, such that we may write $M_{ij}(t) = M_{ij} T \delta(t)$. The force density becomes

$$F_i(\mathbf{R}, t) \quad = \quad M_{ij} T \delta(t) \partial_j \delta(\mathbf{R} - \mathbf{R}_0) \ . \tag{6.3}$$

We call an earthquake with a $\delta(\mathbf{R} - \mathbf{R}_0)$ spatial dependence and a $\delta(t)$ time dependence of the focal force (or the derivatives of the δ-function) an elementary earthquake, produced by an elementary seismic source. The tensor M_{ij} is called the tensor of the seismic moment. Sometimes, it is convenient to use the reduced tensor $m_{ij} = M_{ij}/\rho$, where ρ is the density of the body.

[6] G. G. Stokes, "On the dynamical theory of diffraction", Trans. Phil. Soc. Cambridge **9** 1 (1849) (reprinted in *Math. Phys. Papers*, vol. 2, Cambridge University Press, Cambridge 1883, p. 243).

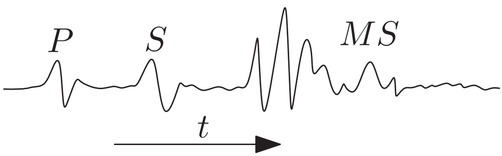

Figure 6.1: Schematic representation of a typical seismogram, with the P and S waves and the main shocks MS; the arrow indicates the flow of the time t.

The force density given by equation (6.3) is the tensorial focal force density for an elementary earthquake.[7] It is a representation of what is called in Seismology the "double-couple" force.[8] We can see that the total force

$$\int d\boldsymbol{R}F_i = \int d\boldsymbol{R}M_{ij}(t)\partial_j\delta(\mathbf{R}-\mathbf{R}_0) = 0 \tag{6.4}$$

is vanishing (due to the presence of the derivatives $\partial_j\delta(\mathbf{R}-\mathbf{R}_0)$) and, also, the total torque of the force

$$\int d\boldsymbol{R}\varepsilon_{ijk}x_jF_k = \int d\boldsymbol{R}\varepsilon_{ijk}x_jM_{kl}(t)\partial_l\delta(\mathbf{R}-\mathbf{R}_0) = 0 \tag{6.5}$$

is vanishing (by an integration by parts, and assuming that the seismic tensor is symmetric; ε_{ijk} is the total antisymmetric tensor of rank three).

The (symmetric) seismic tensor M_{ij} has in general six components. For a shear fault the number of components is reduced to four (by Kostrov representation[9]); in this case it can be deduced from the

[7] B. F. Apostol, "Elastic displacement in a half-space under the action of a tensor force. General solution for the half-space with point forces", J. Elas. **126** 231 (2017); "Elastic waves inside and on the surface of a half-space", Quart. J. Mech. Appl. Math. **70** 289 (2017); *Introduction to the Theory of Earthquakes*, Cambridge International Science Publishing, Cambridge (2017); *The Theory of Earthquakes*, Cambridge International Science Publishing, Cambridge (2017); *Seismology*, Nova, NY (2020).

[8] K. Aki and P. G. Richards, *Quantitative Seismology*, 2nd edition, University Science Books, Sausalito, CA (2009), p. 60, Exercise 3.6.

[9] B. V. Kostrov, "Seismic moment and energy of earthquakes, and seismic flow

recordings of the P and S seismic waves, by imposing the covariance condition.[10] For an explosion the seismic tensor reduces to a scalar ($M_{ij} = -M\delta_{ij}$).

It was realized that the logarithm of the soil displacement measured at Earth's surface is a measure of the strength of the earthquake, called magnitude. Since the displacement is related to the energy E released by the earthquake, this law can be written as

$$\lg E = a + bM_w \ , \tag{6.6}$$

where M_w is the magnitude of the earthquake and a and b are conventional numerical coefficients. This is known as the Gutenberg-Richter law.[11] On the other hand, we can see from equation (6.3) that the magnitude $\overline{M} = \left(M_{ij}^2\right)^{1/2}$ of the seismic tensor is a measure of the energy released in the focus. The relation $E = \overline{M}/2\sqrt{2}$ was established recently.[12] Equation (6.6) can be written as

$$\lg \overline{M} = \left(a + \frac{3}{2}\lg 2\right) + bM_w \ , \tag{6.7}$$

or, by convention,

$$\lg \overline{M} = \frac{3}{2}M_w + 16.1 \ , \tag{6.8}$$

where $\overline{M}$ is measured in *erg* ($1dyn \cdot 1cm$); therefore, $a = 15.65$ and $b = \frac{3}{2}$. This equation is known as the Hanks-Kanamori law.[13]

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of rock", Bull. (Izv.) Acad. Sci. USSR, Earth Physics, **1** 23 (1974) (English translation pp. 13-21); B. V. Kostrov and S. Das, *Principles of Earthquake Source Mechanics*, Cambridge University Press, NY (1988).

[10] B. F. Apostol, "An inverse problem in seismology: derivation of the seismic source parameters from P and S seismic waves", J. Seism. **23** 1017 (2019); "On an inverse problem in elastic wave propagation", Roum. J. Phys. **64** 114 (2019).

[11] B. Gutenberg and C. Richter, "Frequency of earthquakes in California", Bull. Seism. Soc. Am. **34** 185 (1944); "Magnitude and energy of earthquakes", Annali di Geofisica **9** 1 (1956) (Ann. Geophys. **53** 7 (2010)).

[12] B. F. Apostol, "An inverse problem in seismology: derivation of the seismic source parameters from P and S seismic waves", J. Seism. **23** 1017 (2019).

[13] H. Kanamori, "The energy release in earthquakes", J. Geophys. Res. **82** 2981 (1977); T. C. Hanks and H. Kanamori, "A moment magnitude scale", J. Geophys. Res. **84** 2348 (1979).

The magnitude of the seismic moment is determined by using semi-empirical laws, especially its relation with the soil displacement, *i.e.* the solution of the elastic wave equation; synthetic seismograms are produced this way, which are fitted to the recorded seismograms.[14] Usually, the solution of the Stokes problem is used to construct a solution for a couple of forces (with an infinitesimal arm), then another couple is added to avoid free rotations. This is called the double-couple solution.[15] Various other semi-empirical parameters are introduced. The comparison is made with time-frequency windows of the main shock. This procedure involves many arbitrary approximations, or at least, of a very particular nature, and it is not fully in the public domain. It is incorporated in numerical codes, provided by various agencies. The determination of the seismic moment, as well as of all the other parameters of the seismic source from the seismic P and S waves was established recently.[16]

[14] A. M. Dziewonski and D. L. Anderson, "Preliminary reference Earth model", Phys. Earth planet. Inter. **25** 297 (1981); M. L. Jost and R. B. Herrmann, "A student's guide to and review of moment tensors", Seismol. Res. Lett. **60** 37 (1989); D. Giardini, "Moment tensor inversion from mednet data (I). Large worldwide earthquakes of 1990", Geophys. Res. Lett. **19** 713 (1992); F. Bernardi, J. Braunmiller, K. Kradolfer and D. Giardini, "Automatic regional moment tensor inversion in the European-Mediterranean region", Geophys. J. Int. **157** 703 (1995); H. Kawakatsu, "Automated near real-time CMT inversion", Geophys. Res. Lett. **22** 2569 (1995); Z. H. Shomali, "Empirical Green functions calculated from the inversion of earthquake radiation patterns", Geophys. J. Int. **144** 647 (2001); G. Ekstrom, M. Nettles and A. M. Dziewonski, "The global CMT project 2004-2010: centroid-moment tensors for 13,017 earthquakes", Phys. Earth planet. Int. **200-201** 1 (2012).

[15] See, for instance, M. Bath, *Mathematical Aspects of Seismology*, Elsevier, Amsterdam (1968); A. Ben-Menahem and J. D. Singh, *Seismic Waves and Sources*, Springer, NY (1981); K. Aki and P. G. Richards, *Quantitative Seismology*, University Science Books, Sausalito, CA (2009).

[16] B. F. Apostol, "An inverse problem in seismology: derivation of the seismic source parameters from P and S seismic waves", J. Seism. **23** 1017 (2019).

6.2 Primary waves: P and S seismic waves

The equation of the elastic waves in a homogeneous isotropic body (Navier-Cauchy equation[17]) is

$$\ddot{\mathbf{u}} - c_t^2\Delta\mathbf{u} - (c_l^2 - c_t^2) grad\, div\mathbf{u} = \boldsymbol{F} \ , \tag{6.9}$$

where $\mathbf{u}$ is the displacement vector, $c_{l,t}$ are the wave velocities and $\mathbf{F}$ is the force per unit mass (reduced force, force density divided by the density ρ of the elastic body).[18] The velocities are related to the Lame coefficients λ and μ by $c_l = \sqrt{(\lambda+2\mu)/\rho}$, $c_t = \sqrt{\mu/\rho}$. We solve this equation in the infinite space by a particular solution given by the tensorial force

$$F_i = m_{ij}T\delta(t)\partial_j\delta(\boldsymbol{R}) \ ; \tag{6.10}$$

equation (6.9) reads now

$$\ddot{u}_i - c_t^2\Delta u_i - (c_l^2 - c_t^2)\partial_i\partial_j u_j = m_{ij}T\delta(t)\partial_j\delta(\boldsymbol{R}) \ , \tag{6.11}$$

where $m_{ij} = M_{ij}/\rho$. The solution has a near-field part, corresponding to small distances R (comparable to the dimension of the focus) and a far-field part, corresponding to large R. We give here the far-field displacement[19]

$$\begin{aligned} u_i^f = & \frac{T}{4\pi c_t}\frac{m_{ij}x_j}{R^2}\delta'(R-c_t t) + \\ & + \frac{T}{4\pi}\frac{m_{jk}x_i x_j x_k}{R^4}\left[\frac{1}{c_l}\delta'(R-c_l t) - \frac{1}{c_t}\delta'(R-c_t t)\right] \end{aligned} \tag{6.12}$$

for the tensorial force.

In the far-field region the c_l-contribution is the longitudinal (P) wave, while the c_t-contribution is the transverse (S) wave (this can easily

[17] C. L. Navier, Mem. Acad. Sci. t. 7 (1827) (read in Paris Academy in 1821); A. L. Cauchy, "Sur les equations qui expriment les conditions d'equilibre ou les lois du mouvement interieur d'un corps, solide, elastique ou non-elastique", included in *Exercices de Mathematique* (1828) (communicated to the Paris Academy in 1822).

[18] L. Landau and E. Lifshitz, *Course of Theoretical Physics*, vol. 7, *Theory of Elasticity*, Elsevier, Oxford (1986).

[19] B. F. Apostol, "Elastic waves inside and on the surface of a half-space", Quart. J. Mech. Appl. Math. **70** 289 (2017).

be seen by multiplying by x_i and summing over i in equation (6.12)). The scissor-like shape of the δ'-functions appears in seismograms (as shown in Fig. 6.1, with oscillations). We can see that the P and S waves are spherical-shell waves, concentrated on spherical surfaces with radius $R = c_{l,t}t$. We call them primary waves. (In the near-field region the $c_{l,t}$-contributions have not purely longitudinal or purely transverse polarizations).

For an isotropic source $m_{ij} = -m\delta_{ij}$ and the displacements reduce to

$$\boldsymbol{u}^n = \frac{Tm\boldsymbol{R}}{4\pi c_l R^3}\delta(R-c_lt) \ , \ \boldsymbol{u}^f = -\frac{Tm\boldsymbol{R}}{4\pi c_l R^2}\delta'(R-c_lt) \ , \tag{6.13}$$

where $\boldsymbol{u}^n$ is the near-field displacement. We can see that these displacements are purely longitudinal, as expected.

From equation (6.9) we get the energy conservation in the elastic waves.[20] If we multiply equation (6.9) by $\dot{u}_i$ and perform summation over the suffix i, we get the law of energy conservation

$$\begin{gathered}\frac{\partial}{\partial t}\left[\frac{1}{2}\rho\dot{u}_i^2 + \frac{1}{2}\rho c_t^2(\partial_j u_i)^2 + \frac{1}{2}\rho(c_l^2-c_t^2)(\partial_i u_i)^2\right] - \\ -\rho c_t^2\partial_j(\dot{u}_i\partial_j u_i) - \rho(c_l^2-c_t^2)\partial_j(\dot{u}_j\partial_i u_i) = \rho\dot{u}_i F_i \ .\end{gathered} \tag{6.14}$$

According to this equation, the external force $\boldsymbol{F}$ performs a mechanical work $\rho\dot{\boldsymbol{u}}\boldsymbol{F}$ (in the focus) per unit volume and unit time. The corresponding energy is transferred to the waves (the terms in the square brackets in equation (6.14)), which carry it through the space (the terms including the *div* in equation (6.14)). It is worth noting that outside the focal region the force is vanishing. Also, the waves do not exist inside the focal region. Therefore, limiting ourselves to the displacement vector of the waves, we have not access to the mechanical work done by the external force in the focal region. This circumstance arises from the localized character of the focus (which is separated from the body).

Using the decomposition $\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t$, $curl\mathbf{u}_l = 0$, $div\mathbf{u}_t = 0$, equation (6.14) is transformed into

$$\frac{\partial e_{l,t}}{\partial t} + c_{l,t}div\mathbf{s}_{l,t} = 0 \ , \tag{6.15}$$

[20] B. F. Apostol, *Seismology*, Nova (2020).

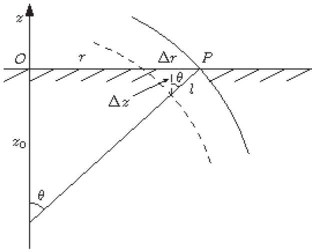

Figure 6.2: Spherical-shell wave intersecting the surface $z = 0$ at P.

where

$$e_{l,t} = \frac{1}{2}\rho\left(\dot{u}^f_{l,ti}\right)^2 + \frac{1}{2}\rho c^2_{l,t}\left(\partial_i u^f_{l,tj}\right)^2 \tag{6.16}$$

is the energy density and

$$s_{l,ti} = -\rho c_{l,t}\dot{u}^f_{l,tj}\partial_i u^f_{l,tj} \; ; \tag{6.17}$$

$c_{l,t}s_{l,ti}$ are energy flux densities per unit time (energy flow). From equation (6.15) we can see that the energy is transported with velocities $c_{l,t}$ (as it is well known). The volume energy $E = \int d\mathbf{R}(e_l + e_t)$ is equal to the total energy flux

$$\begin{gathered}\Phi = -\int dtd\mathbf{R}\left(c_l div\mathbf{s}_l + c_t div\mathbf{s}_t\right) = \\ = -\int dt \oint d\mathbf{S}\left(c_l\mathbf{s}_l + c_t\mathbf{s}_t\right) \; ,\end{gathered} \tag{6.18}$$

i.e. the energy is fully transported by waves.

6.3 Secondary waves: main seismic shock

The primary waves are concentrated on spherical shells with thickness of the order $l = \Delta R = cT \ll R$ (for convenience, we discuss only one primary wave, with velocity c). They propagate from the focus and intersect the free surface of the Earth, placed at height z_0 from the focus. This height defines the position of the epicentre on the surface of the Earth. The wavefront of the primary waves intersect the surface of the Earth along a circular line, with radius r, measured from the epicentre. The intersection line of the primary waves with this surface has a finite spread Δr, which can be calculated from

$$R^2 = r^2 + z_0^2 \, , \; (R+l)^2 = (r+\Delta r)^2 + z_0^2 \; ; \tag{6.19}$$

hence,

$$\Delta r = \frac{2Rl + l^2}{r + \sqrt{r^2 + 2Rl + l^2}} \; ; \tag{6.20}$$

we can see that for $r \to 0$ the width $\Delta r \simeq \sqrt{2z_0 l}$ of the seismic spot on the surface is much larger than the width of the spot for large distances $\Delta r \simeq l$ ($2z_0 \gg l$). We may call the region with the radius $\sqrt{2z_0 l}$ surrounding the epicentre the epicentral region. For values of r not too close to the epicentre we may use the approximation $\Delta r \simeq Rl/r$. A spherical wave intersecting the surface $z = 0$ is shown in Fig. 6.2.

The energy density of the spherical waves is proportional to $1/R^2$. As long as the spherical wave is fully included in the half-space its total energy E_0 is given by the energy density integrated over the spherical shell of radius R and thickness l. If the wave intersects the surface of the half-space, its energy E is given by the energy density integrated over the spherical sector which subtends the solid angle $2\pi(1+\cos\theta)$, where $\cos\theta = z_0/R$ (see Fig. 6.2). It follows $E = \frac{1}{2}E_0\,(1 + z_0/ct)$ for $ct > z_0$. We can see that the energy of the wave decreases by the amount $E_s = \frac{1}{2}E_0\,(1 - z_0/ct)$, $ct > z_0$. This amount of energy is transferred to the surface, which generates secondary waves (according to Huygens principle).

In the seismic spot with width Δr generated on the surface by the far-field primary waves we may expect a reaction of the (free) surface, such as to compensate the force exerted by the incoming spherical waves (boundary force). The waves propagate by transferring motion from particle to particle. The particles on the free surface return the motion to the body. This localized reaction force generates secondary waves, distinct from the incoming, primary spherical waves. The secondary waves can be viewed as waves scattered off the surface (reflected), in the whole half-space, from the small region of contact of the surface seismic spot (a circular line). If the reaction force is strictly limited to the zero-thickness surface (as, for instance, a surface force), it would not give rise to waves, since its source has a zero integration measure. We assume that this reaction appears in a surface layer of thickness Δz ($\Delta z \ll z_0$) and with a surface extension $2\pi r \Delta r$, where it is produced by volume forces. The thickness Δz of the superficial layer activated by the incoming primary wave may depend on R (and r), similarly for the surface spread Δr; for instance, from Fig. 6.2 we have $\Delta z = lz_0/R$. We limit ourselves to an intermediate, limited region of the variable r (R) (*i.e.*, for a region not very close to the epicentre and not extending to infinity), of the order z_0.

We can calculate the displacement $\boldsymbol{u}_s$ in the secondary waves by solving the elastic waves equation, with a force $\boldsymbol{f}$ localized on the surface, in the region where the primary waves intersect the surface of the Earth (the superficial layer of thickness Δz and surface spread $2\pi r \Delta r$). Such a force is usually derived from two potentials χ and $\boldsymbol{h}$, by $\boldsymbol{f} = grad\chi + curl\boldsymbol{h}$ ($div\boldsymbol{h} = 0$). We take these potentials as[21]

$$\chi = \chi_0(r)\delta(z)\delta(r - v_l t) \ , \ \mathbf{h} = \mathbf{h}_0(r)\delta(z)\delta(r - v_t t) \ . \tag{6.21}$$

Equations (6.21) describe wave sources, distributed uniformly along circular lines on the surface, propagating on the surface with constant velocities $v_{l,t}$ and limited to a superficial layer with "zero" thickness and a circular line ("zero" width); their magnitudes $\chi_0(r)$ and $\mathbf{h}_0(r)$ have an approximate m/R-dependence, where m is an order-of-magnitude estimate of the seismic moment m_{ij}. These quantities have a slow variation for $r \sim z_0$ (not very close to the epicentre); for this range of the variable r we may consider χ_0 and $\mathbf{h}_0$ as being constant. The velocities $v_{l,t}$ in equation (6.21) correspond to the velocities $v_{l,t} = dr/dt = c_{l,t}^2 t/r$ resulting from equation (6.19). We see that $v_{l,t}$ are greater than $c_{l,t}$, *i.e.* the primary waves move faster on the surface of the Earth than the secondary waves they generate. In addition, $v_{l,t}$ depend on r and tend to $c_{l,t}$ for large values of the distance r. We make a further simplification and consider them as constant velocities, slightly greater than $c_{l,t}$ (over an intermediate, limited range of variation of r). Also, in the subsequent calculations we consider the origin

[21] B. F. Apostol, "Elastic waves inside and on the surface of a half-space", Quart. J. Mech. Appl. Math. **70** 289 (2017); *Introduction to the Theory of Earthquakes*, Cambridge International Science Publishing, Cambridge (2017); *The Theory of Earthquakes*, Cambridge International Science Publishing, Cambridge (2017); *Seismology*, Nova, NY (2020).

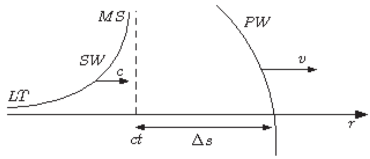

Figure 6.3: Primary wave (PW), moving with velocity v on the Earth's surface, secondary wave (SW), moving with velocity $c < v$, the main shock (MS) and the long tail (LT); the separation between the two wavefronts is $\Delta s = 2(v - c)t$ and the time delay is $\Delta t = (2r/c)(v/c - 1)$, where r is the distance on the surface from the epicentre.

of the time at $r = 0$ (the epicentre) for each primary wave and the associated secondary source. The simplified model of secondary sources introduced here retains the main features of the exact problem, incorporated in the surface localization and propagation of the sources with velocities $v_{l,t}$ greater than wave velocities $c_{l,t}$; on the other hand, by using this model we lose the anisotropy induced by the tensor of the seismic moment and specific details regarding the dependence on the distance. Since the secondary seismic sources are sources moving on the surface we may call the secondary waves produced by these sources "surface seismic radiation".

The potentials given above give rise to the leading contributions to the components of the surface displacement ($z = 0$, in polar cylindrical coordinates)[22]

$$u_{sr} \simeq \frac{\chi_0 \tau_l}{4c_l} \cdot \frac{r}{\left(c_l^2 \tau_l^2 - r^2\right)^{3/2}} \ ,$$

$$u_{s\varphi} \simeq -\frac{h_{0z} \tau_t}{4c_t} \cdot \frac{r}{(c_t^2 \tau_t^2 - r^2)^{3/2}} \ , \tag{6.22}$$

$$u_{sz} \simeq \frac{h_{0\varphi} \tau_t}{4c_t} \cdot \frac{c_t^2 \tau_t^2}{r(c_t^2 \tau_t^2 - r^2)^{3/2}} \ ,$$

where $\tau_{l,t}$ denote times slightly smaller than t. We can see that there exists a horizontal component of the displacement perpendicular to

[22] B. F. Apostol, "Elastic waves inside and on the surface of a half-space", Quart. J. Mech. Appl. Math. **70** 289 (2017).

the propagation direction ($u_{s\varphi}$) and both the r-component and the φ, z-components, which make right angles with the propagation direction, are of the same order of magnitude.[23] For long times ($c_{l,t}\tau_{l,t} \gg r$) the surface displacement goes like

$$u_{sr} \simeq \frac{\chi_0 r}{4c_l^4\tau_l^2} \ , \ u_{s\varphi} \simeq -\frac{h_{0z} r}{4c_t^4\tau_t^2} \ , \ u_{sz} \simeq \frac{h_{0\varphi}}{4c_t^2 r} \ ; \tag{6.23}$$

these equations show that the displacement exhibits a long tail, especially the z-component. It is worth noting the singular character of the solution near the wavefront, where the displacement looks like a wall. The solutions given above are valid for distances r larger than the radius $\sqrt{2z_0 l}$ of the epicentral region, but sufficiently small, such that the thickness $\Delta z = lz_0/R$ of the superficial layer of sources to be comparable to the thickness l of the primary waves. It follows that the maximum of the main shock appears for distances r of the order of the depth z_0. The solutions given by the above equations describe the seismic main shock. Primary and secondary waves, the main shock and the long tail are shown in Fig. 6.3. The P and S seismic waves and the main shock are the main characteristics of a typical seismogram.[24]

Also, secondary waves produced by an internal discontinuity surface, parallel to the free surface of the half-space, have been calculated. The discontinuity surface, excited by primary waves, separates two bodies with distinct elastic properties. Secondary waves propagating above and below the discontinuity surface are refracted and reflected waves. They differ from the refracted and reflected waves produced by a plane wave.

The calculation of a typical seismogram presented here has been extended to other, more complex, situations. For instance, we can consider a structured focus, consisting of a set of pointlike elementary foci, acting at different moments of time. Also, we can consider a moving focus, in which case we have a "seismic radiation". On the

[23] A. E. H. Love, *Some Problems of Geodynamics*, Cambridge University Press, London (1926).

[24] R. D. Oldham, "On the propagation of earthquake motion to long distances", Trans. Phil. Roy. Soc. London **A194** 135 (1900); C. G. Knott, *The Physics of Earthquake Phenomena*, Clarendon Press, Oxford (1908); A. E. H. Love, *Some Problems of Geodynamics*, Cambridge University Press, London (1926).

surface of seas and oceans the abrupt wall of the main shock looks like a tsunami. The focus may be localized on the surface of the Earth, in which case, it is isotropic, we have an explosion or a meteorite. Also, the focus may be localized beneath the bottom of seas and oceans.[25]

The static elastic deformations caused by a tensor force $F_i(\boldsymbol{R}) = M_{ij}\partial_j\delta(\boldsymbol{R}-\boldsymbol{R}_0)$ (equation (6.2)) have been computed for a homogeneous and isotropic half-space.[26] Small crustal quasi-static deformations can be measured by laser techniques from satellites; they release the seismic energy accumulated in the focus. A long intermittence of these deformations may indicate the accumulation of a large amount of seismic energy. Therefore, watching the crustal quasi-static deformations, especially in a presumable epicentral region, may give information about the seismic hazard.

6.4 Main shock

Let a spherical-shell primary wave touch the plane surface of the Earth at the epicentre. The depth of the focus is z_0 and the thickness of the wave is l, of the order of the dimension of the focus. The wave affects a circular disk on Earth's surface with radius d_e, given by $(z_0+l)^2 = z_0^2 + d_e^2$, *i.e.* $d_e \simeq \sqrt{2z_0 l}$ $(l \ll z_0)$. This is the radius of the epicentral region. For instance, for $z_0 = 100km$ and $l = 500m$ we get $d_e = 10km$.

Thereafter, the primary wave propagates along the surface of the Earth as a circular ring with velocity

$$v = \frac{dr}{dt} = \frac{d}{dt}\sqrt{R^2 - z_0^2} = c\frac{R}{r} \ , \tag{6.24}$$

where r is the radius of the ring on the surface of the Earth, $R = ct$ is the distance from the focus and c is the velocity of the spherical-shell primary wave (Fig. 6.2). We can see that v is a non-uniform velocity, which varies from ∞ ($r = 0$, epicentre) to c ($r \longrightarrow \infty$). It

[25]B. F. Apostol, *Seismology*, Nova, NY (2020).

[26]B. F. Apostol, "Elastic displacement in a half-space under the action of a tensor force. General solution for the half-space with point forces", J. Elast. **126** 231 (2017).

is always higher than the elastic wave velocity c. The spot of the primary wave propagates on the surface of the Earth faster than the elastic wave velocity. We look for its effect on the surface of the Earth for distances much larger than the epicentral distance ($r \gg d_e$).

The circular ring of the primary wave on the suface of the Earth has a thickness Δr given by equation (6.20). It varies from the epicentral distance d_e ($r = 0$) to l ($r \longrightarrow \infty$). We may take approximately $\Delta r \simeq \frac{R}{r} l$ for $r \gg d_e$.

In addition, according to Fig. 6.2, the spherical shell of the primary wave has a vertical thickness $\Delta z = \frac{z_0}{R} l$; it varies from l ($r = 0$) to 0 ($r \longrightarrow \infty$). In the circular ring with the cross-section of the order $\Delta r \Delta z$, of the primary wave on the surface of the Earth, volume forces occur, which are sources for secondary waves. For these volume forces the cross-section of the ring should be of the order l^2, *i.e.* $\Delta r \Delta z = \frac{z_0}{r} l^2 \simeq l^2$. It follows that the secondary waves (the main shock) derived above are valid for intermediate distances r, centered on z_0.

A critical quantity in the derivation of the main shock is $\varepsilon = v/c - 1 = \frac{R}{r} - 1$, where $0 < \varepsilon < 1$.[27] For $\varepsilon > 1$ the main shock has not been formed yet, while $\varepsilon \longrightarrow 0$ for $r \longrightarrow \infty$, where the secondary waves disappear (and their expressions given above are not valid anymore). The threshold $\varepsilon = 1$ gives $r = z_0/\sqrt{3}$, which may be taken as an estimate of the spread of the r-region centered on $r = z_0$, where the expressions given here for the secondary waves are valid. For $r = z_0$ we get $\varepsilon = \sqrt{2} - 1$.

The parameter ε governs the time delay between the arrival of the primary wave and the main shock. The primary wave arrives at the time $\tau_p = (r/c)/(1 + \varepsilon)$ at the point r, while the main shock arrives at the same point at the time $\tau_m = (r/c)/(1 - \varepsilon)$. The time delay is $\Delta\tau = 2(r/c)\varepsilon/(1 - \varepsilon^2)$. For $r = z_0$ ($\varepsilon = \sqrt{2} - 1$) we get $\tau_p = 0.7(r/c)$ and $\tau_m = 1.7(r/c)$ (where $r = z_0$); the time delay is $\Delta\tau \simeq r/c$.

We pass now to estimate the magnitude of the main shock u_{ms}. According to equation (6.22) the displacement of the main shock is singular for $c\tau = r$. This singularity is smoothed out by the uncertainty

[27] B. F. Apostol, "Elastic waves inside and on the surface of a half-space", Quart. J. Mech. Appl. Math. **70** 289 (2017); *Seismology*, Nova, NY (2020).

introduced by the parameter ε, such that $c^2\tau^2 - r^2 \longrightarrow c^2\tau^2\varepsilon$. From equation (6.22) we get

$$u_{ms} \simeq \frac{\chi_0}{4c^2}\frac{1}{r\varepsilon^{3/2}} = \frac{\chi_0}{4c^2}\frac{\sqrt{r}(R+r)^{3/2}}{z_0^3} \tag{6.25}$$

for $c\tau = r$, where χ_0 is a generic notation for the magnitudes χ_0 and h_0 of the potentials. Making use of equation (6.21) we get $\chi \simeq \chi_0/\Delta z\Delta r$, such that $\chi_0 = (z_0 l^2/r)\chi$. The potential χ is a generic notation for the potentials χ and $\boldsymbol{h}$, which can be estimated from $\boldsymbol{f} = grad\chi + curl\boldsymbol{h}$ ($div\boldsymbol{h} = 0$), where $\boldsymbol{f}$ is the density of the elastic volume force (per unit mass) generated by the displacement of the primary waves. The estimate is $\chi \simeq cTM/4\pi\rho Rl^3 = M/4\pi\rho Rl^2$, where M is the magnitude of the seismic moment and ρ is the density. On the other hand, the magnitude of the seismic moment is of the order $M \simeq \rho c^2 l^3$, *i.e.* the elastic energy stored in the focus. Therefore, $\chi \simeq c^2 l/4\pi R$ and $\chi_0 \simeq c^2 l^3 z_0/4\pi rR$. Finally, we get

$$u_{ms} \simeq \frac{l^3}{16\pi}\frac{(R+r)^{3/2}}{z_0^2\sqrt{r}R} \ ; \tag{6.26}$$

this estimate is valid for an intermediate range of distances r, say from $z_0/\sqrt{3}$ to $z_0 + (z_0 - z_0/\sqrt{3}) = z_0(2 - 1/\sqrt{3})$. For $r = z_0$ we get u_{ms} of the order l^3/z_0^2.

This displacement should be compared with the displacement produced by the primary wave, which is of the order $u_p \simeq l^2/R$ (equation (6.12)). We can see that $u_{ms} \ll u_p$ (on the average). The damaging effect of the main shock arises from its long duration, which is of the order z_0/c, compared with the duration $\simeq l/c$ of the primary waves.

6.5 Earthquake parameters

The main parameter of an earthquake is the tensor of the seismic moment M_{ij} of the focus. We show here how it can be derived, together with other earthquake parameters, from the measurements of the P and S seismic waves at the surface of the Earth.

For an elementary earthquake the displacement produced by the far-field seismic waves are given by

$$u_i^f = -\frac{T}{4\pi\rho c_t^3}\frac{M_{ij}x_j}{R^2}\delta^{'}(t-R/c_t) - \frac{T}{4\pi\rho}\frac{M_{jk}x_i x_j x_k}{R^4}\cdot \cdot\left[\frac{1}{c_l^3}\delta^{'}(t-R/c_l) - \frac{1}{c_t^3}\delta^{'}(t-R/c_t)\right] \tag{6.27}$$

(equation (6.12). This equation can be decomposed in longitudinal and transverse waves, which are the P wave and the S wave, respectively. In addition, we may replace $\delta^{'}$ by $1/T^2$, such that we get

$$v_{li} = -\frac{1}{4\pi\rho T c_l^3}\frac{M_{jk}x_i x_j x_k}{R^4} \ , \qquad v_{ti} = \frac{1}{4\pi\rho T c_t^3 R^2}\left(\frac{M_{jk}x_i x_j x_k}{R^2} - M_{ij}x_j\right) \tag{6.28}$$

for the amplitudes of these waves. We introduce the unit vector $\boldsymbol{n} = \boldsymbol{R}/R$, where $\boldsymbol{R}$ is the vector from the focus to the observation point. Also, we introduce the notations

$$M_i = M_{ij}n_j \ , \ M_0 = M_{ii} \ , \ M_4 = M_{ij}n_i n_j \ . \tag{6.29}$$

Henceforth, we consider the unit vector $\mathbf{n}$ a known vector. M_0 is the trace of the seismic-moment tensor and M_4 is the quadratic form associated to the seismic-moment tensor, constructed with the unit vector $\mathbf{n}$; we may call it the unit quadratic form of the tensor. The vector $\mathbf{M}$ can be called the "projection" of the tensor along the focus-observation point direction (observation direction). We get easily

$$\mathbf{v}_l = -\frac{1}{4\pi\rho T c_l^3 R}M_4\mathbf{n} \ , \ \mathbf{v}_t = \frac{1}{4\pi\rho T c_t^3 R}(M_4\mathbf{n} - \mathbf{M}) \tag{6.30}$$

and

$$\mathbf{M} = -4\pi\rho T R\left(c_l^3\mathbf{v}_l + c_t^3\mathbf{v}_t\right) \ . \tag{6.31}$$

The displacement in the far-field waves is determined by three independent parameters: the magnitude of the vectors $\mathbf{v}_{l,t}$ (two parameters) and the direction of the transverse vector $\mathbf{v}_t$ (one parameter). Consequently, we may view equations (6.31) as three independent equations for the six unknown components M_{ij} of the seismic moment. Therefore, we have three input data (displacement field $\boldsymbol{v}_{l,t}$) and seven unknowns, consisting of six components of the tensor M_{ij} and the duration T of the seismic activity in the focus. The rest of the parameters entering equations (6.31) (density ρ, distance R, velocities $c_{l,t}$ of the elastic waves) are known (*e.g.*, $\rho = 5g/cm^3$, $c_l = 7km/s$, $c_t = 3km/s$).

We note the consistency (compatibility) relation $M^2 > M_4^2$, derived from $v_t^2 > 0$ ($v_{l,t}$ denote the magnitudes of the vectors $\mathbf{v}_{l,t}$).

6.6 Kostrov representation

The earthquakes are produced by shear forces acting in a (localized) fault. The explosions are produced by anisotropic dilatation occurring in a (localized) focus. A hybrid mechanism, which would involve both a shearing slip in a fault and a dilatation (or compression) is impossible. This follows from the definition of the two processes, and it was also proved formally.[28]

We assume that the fault focal region includes two plane-parallel surfaces, each with a (small) area S, separated by a (small) distance d, sliding against one another. The focal area is determined by two lengths $l_{1,2}$, $S = l_1 l_2$. In general, the lengths l_1, l_2, d are distinct; in order to ensure the compatibility with the localization provided by the δ-function (used in deriving the seismic waves), we assume $l_1 = l_2 = d = l$. The fault is determined by its normal $\boldsymbol{s}$, *i.e.* the unit vector perpendicular to the fault surface. The slip along the surface of the fault is characterized by a unit vector $\boldsymbol{a}$, such that $\boldsymbol{sa} = 0$. The torque of the elastic forces acting upon a localized fault has been computed.[29] It provides a seismic moment given by

$$M_{ij} = 2\mu V \left(s_i a_j + a_i s_j\right) \quad , \tag{6.32}$$

where μ is the Lame coefficient and $V = l^3$ is the volume of the faulting

[28] B. F. Apostol, *Seismology*, Nova, NY (2020).

[29] B. F. Apostol, "An inverse problem in seismology: derivation of the seismic source parameters from P and S seismic waves", J. Seismol. **23** 1017 (2019).

focus. This is the Kostrov relation;[30] (it is a vectorial, or dyadic, representation of the seismic moment). We note the invariant $M_0 = M_{ii} = 0$, which tells that the seismic moment in this representation is a traceless tensor.[31] In addition, we note the relations $M_4^0 = M_{ij}s_i s_j = 0$ and $M_i^0 = M_{ij}s_j = 2\mu S u^0 a_i$; the former relation shows that the quadratic form associated to the seismic moment in the focal region is degenerate (it is represented by a conic), while the latter relation shows that the "force" in the focal region is directed along the focal displacement. The relations $M_0 = 0$ and $M_4^0 = 0$ reduce the number of independent components of the tensor M_{ij} from six to four.

It is worth noting an uncertainty (indeterminacy) of the dyadic construction of the seismic-moment tensor (equation (6.32)). We can see from equation (6.32) that the seismic moment is invariant under the inter-change $\mathbf{s} \longleftrightarrow \mathbf{a}$. This means that from the knowledge of the seismic moment M_{ij} we cannot distinguish between the two orthogonal vectors $\mathbf{s}$ and $\mathbf{a}$ (fault direction and fault slip). Another symmetry of the seismic moment given by equation (6.32) is the simultaneous change of sign of the two vectors $\boldsymbol{s}$ and $\boldsymbol{a}$. The symmetry $\mathbf{s} \longleftrightarrow \mathbf{a}$ shows that any fault is associated, in fact, with another orthogonal fault.[32]

Making use of equations (6.18) and (6.30), we get the energy of the elastic waves

$$E = \Phi = \frac{4\pi\rho}{T} R^2 \left(c_l v_l^2 + c_t v_t^2 \right) \ ; \tag{6.33}$$

this relation gives the energy released by the earthquake in terms of the displacement measured in the far-field region and the (short) duration of the earthquake. From equations (6.30) we get the relation

$$E = \frac{1}{4\pi\rho c_t^5 T^3} \left[M^2 - \left(1 - c_t^5/c_l^5\right) M_4^2 \right] \tag{6.34}$$

between energy and the seismic moment. On the other hand, the Kostrov representation allows the estimation of the mechanical work

[30] B. V. Kostrov, "Seismic moment and energy of earthquakes, and seismic flow of rock", Bull. (Izv.) Acad. Sci. USSR, Earth Physics, **1** 23 (1974) (English translation p. 13); B. V. Kostrov and S. Das, *Principles of Earthquake Source Mechanics*, Cambridge University Press, NY (1988).

[31] This particularity gives access to the near-field waves (B. F. Apostol, *Seismology*, Nova (2020).

[32] B. F. Apostol, *Seismology*, Nova, NY (2020).

$W=\mu V$ done by the elastic forces in the fault. (We can see that the mechanical work done in the focal region is of the order of the elastic energy stored in the focal region, as expected). By equating these two energies, we get

$$\mu V=\frac{4\pi\rho}{T}R^2\left(c_l v_l^2+c_t v_t^2\right) \quad , \tag{6.35}$$

or

$$T=\frac{4\pi}{c_t^2 V}R^2\left(c_l v_l^2+c_t v_t^2\right) \quad . \tag{6.36}$$

By using equation (6.36) we can eliminate the unknown T from equations (6.31), which include now only the seismic moment. We note that the volume V is given by

$$M_{ij}^2=8\mu^2V^2 \tag{6.37}$$

(from equation (6.32)).
Also, we note the representation

$$u_{ij}^0=\frac{1}{2}\left(s_i a_j+a_i s_j\right)=\frac{1}{4\mu V}M_{ij} \tag{6.38}$$

for the focal strain, which follows immediately from the definition of the elastic strain and the considerations made above on the geometry of the focal region. This equation relates the focal strain to the seismic moment; it may be used for assessing the accumulation rate of the seismic moment from measurements of the surface strain rate.[33]

6.7 Determination of the seismic moment and source parameters

Making use of the reduced moment $m_{ij}=M_{ij}/2\mu V$ and $m_i=M_i/2\mu V$ $=M_{ij}n_j/2\mu V$, equation (6.32) leads to

$$s_i(\mathbf{na})+a_i(\mathbf{ns})=m_i \ ; \tag{6.39}$$

[33] S. N. Ward, "A multidisciplinary approach to seismic hazard in southern California", Bull. Seism. Soc. Am. **84** 1293 (1994); J. C. Savage and R. W. Simpson, "Surface strain accumulation and the seismic moment tensor", Bull. Seism. Soc. Am. **87** 1345 (1997).

using equations (6.31) and (6.35) the components m_i of the reduced moment are given by

$$m_i = -\frac{T^2}{2R} \cdot \frac{c_l^3 v_{li} + c_t^3 v_{ti}}{c_l v_l^2 + c_t v_t^2} \ . \tag{6.40}$$

We solve here the equations (6.39) for the unit vectors **a** and **s**, subject to the conditions

$$s_i^2 = a_i^2 = 1 \ , \ \ s_i a_i = 0 \ . \tag{6.41}$$

Since $M_0 = 0$ and $M^2 > M_4^2$, we have $m_0 = m_{ii} = 0$ and $m^2 > m_4^2$ (where $m_4 = m_{ij} n_i n_j$ and $m^2 = m_i^2$). From equation (6.40) we have $m_i < 0$. The compatibility condition $m^2 > m_4^2$ can be checked immediately from equation (6.40) (it arises from $v_t^2 > 0$). We write equations (6.39) as

$$\alpha \mathbf{s} + \beta \mathbf{a} = \mathbf{m} \ \ , \tag{6.42}$$

where we introduce two new notations $\alpha = (\mathbf{na})$ and $\beta = (\mathbf{ns})$. We assume that the vectors **s**, **a** and **n** lie in the same plane, *i.e.*

$$\beta \mathbf{s} + \alpha \mathbf{a} = \mathbf{n} \ . \tag{6.43}$$

This condition determines the system of equations and ensures the covariance of the solution; it is the covariance condition. From equations (6.42) and (6.43) we get

$$2\alpha\beta = m_4 \ , \ \ \alpha^2 + \beta^2 = m^2 = 1 \ . \tag{6.44}$$

The equality $m^2 = 1$ (covariance condition) has important consequences; it implies $M^2 = (2\mu V)^2$, such that we can write the seismic moment from equation (6.32) as

$$M_{ij} = M \left(s_i a_j + a_i s_j \right) \ ; \tag{6.45}$$

it follows the magnitude of the seismic moment $\left({M_{ij}}^2 \right)^{1/2} = \sqrt{2} M$;[34] M is the magnitude of the "projection" of the seismic-moment tensor along the observation radius. In addition, from $E = W = \mu V$ (equation (6.28)) we have $E = M/2 = \left({M_{ij}}^2 \right)^{1/2} / 2\sqrt{2}$. The magnitude

[34] See, for instance, P. G. Silver and T. H. Jordan, "Optimal estimation of the scalar seismic moment", Geophys. J. R. Astr. Soc. **70** 755 (1982).

$\left({M_{ij}}^2\right)^{1/2} = \sqrt{2}M = 2\sqrt{2}E$ may be used in the Gutenberg-Richter (Hanks-Kanamori) relation $\lg\left({M_{ij}}^2\right)^{1/2} = 1.5M_w + 16.1$, which defines the magnitude M_w of the earthquake; in terms of the earthquake energy this relation becomes $\lg E = 1.5(M_w - \lg 2) + 16.1$ (where $\lg 2 \simeq 0.3$). We note that an error of an order of magnitude in the seismic moment (M, E, $\left({M_{ij}}^2\right)^{1/2}$) induces an error $\simeq 0.3$ in the magnitude M_w. We introduce the notation $\overline{M} = \left({M_{ij}}^2\right)^{1/2}$ for the magnitude of the seismic-moment tensor ($\overline{M} = \sqrt{2}M$).
Further, from equation (6.40), the equality $m^2 = 1$ can be written as

$$\frac{T^4}{4R^2} \cdot \frac{c_l^6 v_l^2 + c_t^6 v_t^2}{\left(c_l v_l^2 + c_t v_t^2\right)^2} = 1 \quad , \tag{6.46}$$

which gives the duration T in terms of the displacements $v_{l,t}$ measured at distance R. Inserting T in equation (6.36), we get

$$V^2 = \frac{8\pi^2 R^3}{c_t^4}\left(c_l v_l^2 + c_t v_t^2\right)\left(c_l^6 v_l^2 + c_t^6 v_t^2\right)^{1/2} \tag{6.47}$$

and the magnitude of the seismic moment and the energy of the earthquake

$$\begin{aligned} M &= 2E = 2\mu V = \\ &= 2\pi\rho(2R)^{3/2}\left(c_l v_l^2 + c_t v_t^2\right)^{1/2}\left(c_l^6 v_l^2 + c_t^6 v_t^2\right)^{1/4} \end{aligned} \tag{6.48}$$

in terms of the displacements $v_{l,t}$ measured at distance R. In addition, by eliminating R^2 between equations (6.36) and (6.46), we can express the focal volume as

$$V = \frac{\pi T^3}{c_t^2} \cdot \frac{c_l^6 v_l^2 + c_t^6 v_t^2}{c_l v_l^2 + c_t v_t^2} \quad . \tag{6.49}$$

The solutions of the system of equations (6.44) are given by

$$\alpha = \sqrt{\frac{1 + \sqrt{1 - m_4^2}}{2}} \ , \ \beta = sgn(m_4)\sqrt{\frac{1 - \sqrt{1 - m_4^2}}{2}} \tag{6.50}$$

and $\alpha \longleftrightarrow \pm\beta$, $\alpha, \beta \longleftrightarrow -\alpha, -\beta$. Making use of equations (6.40) and (6.46), the parameters m_i and m_4 are given by

$$m_i = -\frac{c_l^3 v_{li} + c_t^3 v_{ti}}{\left(c_l^6 v_l^2 + c_t^6 v_t^2\right)^{1/2}} \ , \ m_4 = -\frac{c_l^3 (\mathbf{v}_l \mathbf{n})}{\left(c_l^6 v_l^2 + c_t^6 v_t^2\right)^{1/2}} \ . \tag{6.51}$$

Finally, we get the vectors

$$\begin{aligned} \mathbf{s} &= \tfrac{\alpha}{\alpha^2-\beta^2}\mathbf{m} - \tfrac{\beta}{\alpha^2-\beta^2}\mathbf{n} \ , \\ \mathbf{a} &= -\tfrac{\beta}{\alpha^2-\beta^2}\mathbf{m} + \tfrac{\alpha}{\alpha^2-\beta^2}\mathbf{n} \ ; \end{aligned} \tag{6.52}$$

from equations (6.42) and (6.43); these solutions are symmetric under the operations $\mathbf{s} \longleftrightarrow \mathbf{a}$ ($\alpha \longleftrightarrow -\beta$) and $\mathbf{s} \longleftrightarrow -\mathbf{a}$ ($\alpha \longleftrightarrow \beta$, or $\alpha, \beta \longleftrightarrow -\alpha, -\beta$). The seismic moment given by equation (6.45) is determined up to these symmetry operations. We can see that the seismic-moment tensor given by equation (6.45) is determined by M (equation (6.48)) and the vectors $\mathbf{s}$ and $\mathbf{a}$ given by equations (6.52), with the coefficients α, β given by equations (6.50); the vector $\mathbf{n}$ is known and the vector $\mathbf{m}$ and the scalar m_4 are given by the experimental data (equations (6.51)). Equations (6.52) are manifestly covariant.

Finally, by making use of equations (6.52) in equation (6.45) we get the solution for the seismic moment

$$M_{ij} = \frac{M}{1-m_4^2}\left[m_i n_j + m_j n_i - m_4 \left(m_i m_j + n_i n_j\right)\right] \ , \tag{6.53}$$

where M is given by equation (6.48) and m_i, m_4 are given by equations (6.51); the focal strain is $u_{ij}^0 = M_{ij}/2M$ (equation (6.38)). In equation (6.53) there are only three independent components of the seismic tensor, according to the equations $m_{ij}n_j = m_i$ ($m_{ij} = M_{ij}/M$): the vectors $\mathbf{n}$ and $\mathbf{m}$ are known (equation (6.51)) from experimental data, such that these equations can be viewed as three conditions imposed upon the six components M_{ij}. Also, we can see that there exist only three independent components of the seismic tensor M_{ij} from the conditions $M_0 = M_{ii} = 0$, $M_{ij}s_j s_i = 0$ (or $M_{ij}a_i a_j = 0$) and $m_i^2 = 1$. The later equality arises from the covariance condition; this equality and the energy conservation determine the duration T of the earthquake, the volume V of the focal region and the magnitude parameter M of the seismic moment.

6.8 Isotropic seismic moment

An isotropic seismic moment $M_{ij} = -M\delta_{ij}$ is an interesting particular case, since it can be associated with seismic events caused by explosions.[35] In this case the transverse displacement is vanishing ($\mathbf{u}_t^f = 0$), $\mathbf{M} = -M\mathbf{n}$, $M_4 = -M$ and $\mathbf{v}_l = (R/c_l T)\mathbf{u}_l^n$ (equations (6.30)); from equations (6.31) and (6.28) we get

$$\mathbf{M} = -4\pi\rho T R c_l^3 \mathbf{v}_l \ , \ \ E = \frac{4\pi\rho R^2}{T} c_l v_l^2 \tag{6.54}$$

we can see that $\mathbf{v}_l\mathbf{n} > 0$ corresponds to $M > 0$ (explosion), while the case $\mathbf{v}_l\mathbf{n} < 0$ corresponds to an implosion. The focal zone is a sphere with radius of the order l, and the vectors $\mathbf{s}$ and $\mathbf{a}$ are equal ($\mathbf{s} = \mathbf{a}$) and depend on the point on the focal surface; the magnitude of the focal displacement is $u^0 = l$. The considerations made above for the geometry of the focal region lead to the representation

$$M_{ij} = -2V(2\mu + \lambda)\delta_{ij} = -2\rho c_l^2 V \delta_{ij} \ \ , \tag{6.55}$$

where $V = Sl$ denotes the focal volume and S is the area of the focal region. Similarly, the energy is $E = W = \frac{1}{2}M$ ($M > 0$), such that, making use of equations (6.54), we get $c_l T = \sqrt{2Rv_l}$,

$$M = 2\pi\rho c_l^2 (2Rv_l)^{3/2} = 2\rho c_l^2 V \ , \tag{6.56}$$

and the focal volume $V = \pi(2Rv_l)^{3/2}$. These equations determine the seismic moment and the volume of the focal region from the displacement v_l measured at distance R. A superposition of shear faulting and isotropic focal mechanisms (hybrid seismic mechanism) cannot be resolved, because the longitudinal displacement $\mathbf{v}_l$ includes indiscriminately contributions from both mechanisms.[36]

6.9 Qualitative results

We can summarize the results as follows. Making use of the longitudinal displacement $\mathbf{v}_l$ and the transverse displacement $\mathbf{v}_t$, measured

[35] See, for instance, S. E. Minson and D. S. Dreger, "Stable inversions for complete moment tensors", Geophys. J. Int. **174** 585 (2008).

[36] B. F. Apostol, *Seismology*, Nova, NY (2020).

on Earth's surface, we compute the magnitude parameter M from equation (6.48) and the vector $\mathbf{m}$ and the scalar m_4 from equation (6.51); then, from equation (6.53) we get the seismic moment M_{ij}. The energy released by the earthquake is $E = M/2$ (μV) and an estimate of the focal volume is given by $V = M/2\rho c_t^2$. An estimation of the duration T of the earthquake is provided by equation (6.46). The focal slip is of the order $V^{1/3}$ and the focal strain is of the order $M_{ij}/2M$ (equation (6.38)). From the magnitude $\left(M_{ij}{}^2\right)^{1/2} = \sqrt{2}M$ of the seismic moment we may estimate the magnitude M_w of the earthquake by means of the Gutenberg-Richter relation. A similar procedure holds for an isotropic seismic moment.

Making use of $\mathbf{m}$ and m_4 in equations (6.52) we compute the normal $\mathbf{s}$ to the fault plane and the unit slip vector $\mathbf{a}$ in the fault plane; the quadratic form associated to the seismic moment is a degenerate hyperboloid which reduces to a hyperbola in the $(\mathbf{s}, \mathbf{a})$-plane with asymptotes along the vectors $\mathbf{s}$ and $\mathbf{a}$. This hyperbola is tighter (closer to the origin) for higher M.

It is convenient to have an estimation of the order of magnitude of the various quantities introduced here. To this end we use a generic velocity c for the seismic waves and a generic vector $\mathbf{v}$ for the displacement in the far-field seismic waves. Equation (6.46) (which is $m^2 = 1$) gives $cT \simeq \sqrt{2Rv}$, which provides an estimate of the duration of the earthquake in terms of the displacement measured at distance R. The focal volume can be estimated from equation (6.35) as $V \simeq \pi\left(2Rv\right)^{3/2} \simeq \pi(cT)^3$, as expected (dimension l of the focal region of the order cT; the rate of the focal slip is $l/T \simeq c$). Also, from equation (6.48) we have the energy $E \simeq \mu V \simeq M/2 \simeq 2\rho c^2 V$, where M is related to the magnitude $\left(M_{ij}^2\right)^{1/2} = \sqrt{2}M$ of the seismic moment (and the magnitude of the vector $M_{ij}n_j$). From equation (6.38) we get a focal strain of the order unity, as expected. In addition, we can see the relationships

$$\lg E = \lg M + const = \lg V + const = \frac{3}{2}\lg(vR) + const \ , \qquad (6.57)$$

which give a justification for the original identification $\lg(vR) = M_w + const$ and for the coefficient $3/2$ (or $\lg v = M_w + const$ for the same distance R).[37]

6.10 Moment magnitude. Local magnitude

The seismic-moment magnitude M_w is defined by the Hanks-Kanamori relationship[38]

$$\lg\left(M_{ij}{}^2\right)^{1/2}=\frac{3}{2}M_w+16.1\ ,\tag{6.58}$$

where $\overline{M}=\left(M_{ij}{}^2\right)^{1/2}$ is the magnitude of the seismic-moment tensor, measured in erg ($1erg=1dyn\cdot 1cm$) and lg is the decimal logarithm. It is related to the magnitude M of the vector $M_i=M_{ij}n_j$ and the earthquake energy E by the relations $\overline{M}=\sqrt{2}M=2\sqrt{2}E$, where $\boldsymbol{n}$ is the unit vector from the focus to the observation point.[39] Therefore, equation (6.58) can also be written as

$$\lg E=\frac{3}{2}M_w+15.65\ .\tag{6.59}$$

We can see that the seismic-moment magnitude is a measure of the energy released by the earthquakes.

On the other hand, the magnitude of the seismic moment tensor is related to the amplitudes of the P and S seismic waves $v_{l,t}$ by

$$\overline{M}=8\pi\rho R^{3/2}\left(c_l v_l^2+c_t v_t^2\right)^{1/2}\left(c_l^6 v_l^2+c_t^6 v_t^2\right)^{1/4}\ ,\tag{6.60}$$

where ρ is the density of the (homogeneous and isotropic) body, R is the distance from the focus to the observation point and $c_{l,t}$ are the velocities of the longitudinal (l) elastic wave and the transverse (t) elastic wave (equation (6.48)). It follows that we can compute

[37] B. Gutenberg and C. F. Richter, "Frequency of earthquakes in California", Bull. Seism. Soc. Am. **34** 185 (1944).

[38] H. Kanamori, "The energy release in earthquakes", J. Geophys. Res. **82** 2981 (1977); T. C. Hanks and H. Kanamori, "A moment magnitude scale", J. Geophys. Res. **84** 2348 (1979); see also B. Gutenberg and C. Richter, "Frequency of earthquakes in California", Bull. Seism. Soc. Am. **34** 185 (1944); "Magnitude and energy of earthquakes", Annali di Geofisica **9** 1 (1956) (Ann. Geophys. **53** 7 (2010)).

[39] B. F. Apostol, "An inverse problem in seismology: derivation of the seismic source parameters from P and S seismic waves", J. Seismol. **23** 1017 (2019).

the magnitude M_w, by using equations (6.58) and (6.60), from the measurements of the amplitudes $v_{l,t}$ of the seismic waves. The parameters entering these equations can be taken as the mean density of the earth $\rho = 5g/cm^3$ and the elastic waves velocities $c_l = 7km/s$, $c_t = 3km/s$. Obviously, such a determination of the magnitude is affected by errors.

This is why it is convenient to use an approximate formula

$$\overline{M} = 8\pi\rho c^2 \left(Rv\right)^{3/2} \tag{6.61}$$

for equation (6.60), where $c = 5km/s$ is a mean value of the elastic waves velocity and v is a mean value of the amplitude of the elastic waves. Such an approximation implies also a duration T of the seismic activity in the focus given by

$$cT = (2Rv)^{1/2} \tag{6.62}$$

and a volume of the focal region $V \simeq \pi(cT)^3$ (equations (6.36) and (6.46)), such that we may also write

$$\overline{M} = 2\sqrt{2}\rho c^2 V \tag{6.63}$$

and $E = \rho c^2 V$. By means of these approximate formulae we have immediately a direct access to the focal volume and the duration of the seismic activity in the focus.

Making use of equations (6.58) and (6.61), we get

$$\begin{aligned} \lg \overline{M} &= \tfrac{3}{2} \lg (Rv) + \lg(8\pi\rho c^2) = \\ &= \tfrac{3}{2} M_w + 16.1 \ , \end{aligned} \tag{6.64}$$

or

$$\lg v + \lg R = M_w + 1.8 \tag{6.65}$$

for $\rho = 5g/cm^3$ and $c = 5km/s = 5{\times}10^5 cm/s$ $(8\pi\rho c^2 = 10^{13.54} erg/cm^3)$. We can see that $\lg v + \lg R + const$ can be taken as a measure of the seismic-moment magnitude. Since it depends on R, it is called local magnitude, denoted by M_l. We define it as

$$M_l = \lg v + \lg R - 4.8 = M_w - 3 \ ; \tag{6.66}$$

the constant -4.8 is chosen such that we have $M_l = 0$ for $M_w = 3$; this corresponds to $v = 10^{-2.2}cm$ and $R = 100km = 10^7 cm$. In equation (6.66) the displacement v and the distance R are measured in cm. Equation (6.66) is valid only for the approximate estimation of the moment magnitude M_w, which follows by using $\overline{M} \simeq 8\pi\rho c^2 (Rv)^{3/2}$ in equation (6.58), instead of $\overline{M}$ given by equation (6.60).[40]

Equation (6.66) is similar to the great variety of local magnitude scales in use. In particular, the original Richter local magnitude[41] is defined as

$$M_R = \lg v - 2.48 + 2.76 \lg \Delta \ , \tag{6.67}$$

where v is measured in m and Δ is the epicentral distance measured in km; it is calibrated to $M_L = 0$ for $v = 10^{-3}m$ and $\Delta = 100km$ (v is measured in m and Δ is measured in km in equation (6.67)).[42] For shallow earthquakes the epicentral distance $\Delta = \sqrt{R^2 - h^2}$ is approximately equal to R, because the depth of the focus h is much smaller than R.

Task #5 of Practical Seismology is to estimate a mean amplitude v of the P and S seismic waves from recordings and use equations given above to get an estimate of the duration T of the seismic activity in the focus, the focal volume V, the magnitude of the seismic moment $\overline{M}$, the energy of the earthquake E, the moment magnitude M_w and the local magnitude M_l.

Earth is not a homogeneous and isotropic elastic medium; it includes inhomogeneities, which would cast doubts upon the validity of the above derivations. There exist a few inhomogeneities with large dimensions; they affect the direction and the velocity of the elastic waves. This is why, the position of the focus and the epicentre are determined only approximately. The dimensions of the most numerous inhomogeneities are small. They affect the content of the waves in the spherical-shell waves with small wavelengths. Consequently, the seismic waves exhibit a small-amplitude dispersion with short wavelengths (short period of oscillation). This dispersion is visible on seismograms.

[40] B. F. Apostol, "On the local magnitude scale of earthquakes", J. Theor. Phys. **329** (2021).

[41] C. F. Richter, *Elementary Seismology*, Freeman, San Francisco (1958).

[42] T. Lay and T. C. Wallace, *Modern Global Seismology*, Academic, San Diego (1995).

It is compounded with the focus-structure effects and seismographs' eigenoscillations. In estimating the amplitude of the displacement from seismograms we should use the envelope of the oscillations. It may happen that the amplitude is not the same on the two sides of the oscillations; in that case we should use either the maximum amplitude, or a mean amplitude. It is more difficult to establish a direct connection between the displacement in the mainshock and the characteristics of the focal region. In any case, the inhomogeneities structure is more or less uniformly distributed, which makes acceptable the conventional definitions of the earthquakes characteristics, in particular the magnitude. Also, the logarithmic scale in the definition of the magnitude allows for an appreciable reduction of the effect of the large variations.

7 Earthquake Parameters

7.1 Introduction

The determination of the seismic source and earthquake parameters is a basic problem in seismology.[1] This problem is currently solved, almost in real time, by various international and national agencies. The Institute of Earth's Physics at Magurele provides such information for Vrancea earthquakes. The main quantity envisaged by these computations is the seismic-moment tensor and earthquake's magnitude. The solution is obtained by numerically fitting synthetic seismograms to various waveforms recorded at Earth's surface.

Recently, a new method was put forward for determining these parameters from local seismic recordings of the ground displacement produced by the P and S seismic waves at Earth's surface.[2] This theory has no fitting parameters. The components of the force density

[1] A. M. Dziewonski abd D. L. Anderson, "Preliminary reference earth model", Phys. Earth planet. Inter. **25** 297 (1981); S. A. Sipkin, "Estimation of earthquake source parameters by the inversion of waveform data: synthetic waveforms", Phys. Earth planet. Inter. **30** 242 (1982); H. Kawakatsu, "Automated near real-time CMT inversion", Geophys. Res. Lett. **22** 2569 (1995); S. Honda and T. Seno, "Seismic moment tensors and source depths determined by the simultaneous inversion of body and surface waves", Phys. Earth plan. Int. **57** 311 (1989), and References therein; B. Romanowicz, "Inversion of surface waves: a review", in *International Handbook of Earthquake and Engineering Seismology*, vol. 81A, eds. W. Lee, P. Jennings, C. Kisslinger and H. Kanamori, Academic Press, NY (2002); G Ekstrom, M. Nettles and A. M. Dziewonski, "The global CMT project 2004-2010: centroid-moment tensors for 13,017 earthquakes", Phys. Earth Planet. Int. **200-201** 1 (2012); M. Vallee, "Source time function properties indicate a strain drop independent of earthquake depth and magnitude", Nature Commun. doi: 10.1038/mcomms3606 (2013).

[2] B. F. Apostol, "An inverse problem in Seismology: determination of the seismic source parameters from P and S seismic waves", J. Seism. **23** 1017 (2019).

in a seismic focus localized at $\boldsymbol{R} = 0$, with a seismic activity lasting a short time T at the initial moment $t = 0$, is given by

$$f_i = M_{ij} T \delta(t) \partial_j \delta(\mathbf{R}) \ , \tag{7.1}$$

where i, $j = 1$, 2, 3 denote cartesian components and M_{ij} is the (symmetric) tensor of the seismic moment.[3] The Dirac delta functions in equation (7.1) define an elementary earthquake, *i.e.* an earthquake with a localized focus and a short duration. The equation of the elastic waves in a homogeneous, isotropic body has been solved for this force,[4] and the static deformations produced by it in a homogeneous, isotropic half-space have been computed.[5] A homogeneous, isotropic elastic half-space with a plane surface is used as a model for Earth in the seismic regions of interest. It was shown that the force density given by equation (7.1) generates in the far-field region two spherical-shell waves, identified as the P (primary, longitudinal) and S (secondary, transverse) seismic waves. In addition, these primary waves produce wave sources on Earth's surface, with a cummulative elastic energy, which generate secondary waves; the wavefront of the secondary waves has a wall-like profile on Earth's surface, which is the main shock of the earthquakes. A highly simplified sketch of a typical seismogram displaying these features is shown in Fig. 7.1. The theory discussed herein makes use of the amplitudes of the P and S waves computed previously.

Since the theory of determining the tensor M_{ij} may appear as being too technical,[6] we provide here a direct, practical and operative procedure of applying this theory. In particular, the compatibility of the

[3] M. Bath, *Mathematical Aspects of Seismology*, Elsevier, Amsterdam (1968); A. Ben-Menahem and J. D. Singh, *Seismic Waves and Sources*, Springer, NY (1981); A. Udias, *Principles of Seismology*, Cambridge University Press, Cambridge (1999); K. Aki and P. G. Richards, *Quantitative Seismology,* University Science Books, Sausalito, CA (2009); B. F. Apostol, *Introduction to the Theory of Earthquakes*, Cambridge International Science Publishing, Cambridge (2017); *The Theory of Earthquakes*, Cambridge International Science Publishing, Cambridge (2017); *Seismology*, Nova, NY (2020).

[4] B. F. Apostol, "Elastic waves inside and on the surface of a half-space", Quart. J. Mech. Appl. Math. **70** 289 (2017).

[5] B. F. Apostol, "Elastic displacement in a half-space under the action of a tensor force. General solution for the half-space with point forces", J. Elast. **126** 231 (2017).

[6] B. F. Apostol, *loc. cit.*

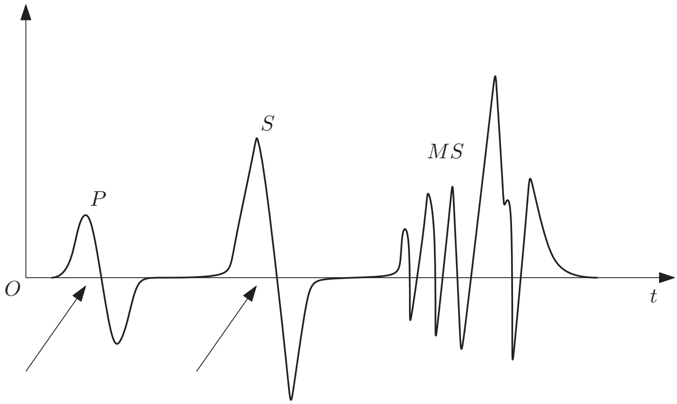

Figure 7.1: Sketch of a typical seismogram, displaying the P- and S-waves and the main shock MS. The arrows indicate the "same side" of the P- and S-waves discussed in text.

input data and the optimization of the errors are discussed. The procedure gives the seismic-moment tensor, the earthquake energy and magnitude, the orientation of the fault and the direction of the tectonic slip, the duration of the focal seismic activity and the dimension of the focal region (fault). The results are exemplified on two Vrancea earthquakes.

7.2 Theory

The basic equations used in this paper relate (algebrically) the longitudinal displacement $\mathbf{v}_l$ (P wave) and the transverse displacement $\mathbf{v}_t$ (S wave), measured at a local site on Earth's surface, to the seismic-moment tensor M_{ij} and the duration T of the focal seismic activity.[7] We assume that the other ingredients entering these relations, like Earth's density and wave velocities, are known. Also, we assume that the position of the focus is known, such that we know the unit vector

[7] B. F. Apostol, "Elastic waves inside and on the surface of a half-space", Quart. J. Mech. Appl. Math. **70** 289 (2017).

n from the focus to the origin of the local frame. Consequently, the data include one parameter of the longitudinal displacement (its magnitude) and two parameters of the transverse displacement; this makes three known parameters. In general, the seismic-moment tensor M_{ij} has six components, which, together with the duration T, make seven unknowns. However, for a fault, the Kostrov representation holds for the seismic-moment tensor,[8] which reduces the number of components from six to four; the energy conservation[9] relates, in fact, one of these components with the earthquake duration, such that the seismic moment has only three independent components for a fault. It follows that we are left with four unknowns and three known parameters (equations). We need a fourth equation in order to solve the problem (*i.e.*, in order to determine the seismic-moment tensor). The fourth equation is provided by the covariance condition, which determines the problem.

7.3 Initial input. Data compatibility

We use a local reference frame with axes, denoted by 1, 2, 3, correspondig to the directions North-South, West-East and the local vertical, respectively. Let θ_0 and φ_0 be the latitude and the longitude of the origin of this local frame. We assume that the latitude θ_E and the longitude φ_E of the epicentre are also known. Then, we determine immediately the coordinates of the epicentre

$$x_1 = -R_0\theta \ , \ x_2 = R_0 \cos\theta_E \cdot \varphi \ , \tag{7.2}$$

where $\theta = \theta_E - \theta_0$, $\varphi = \varphi_E - \varphi_0$ (in radians, *e.g.* $\theta = \theta^\circ \cdot \frac{\pi}{180}$) and $R_0 = 6370km$ is Earth's mean radius. Usually, the depth H of the focus is also given by the seismic measurements, such that we might know the unit vector **n** directed from the focus to the origin of the local frame. Indeed, the vector directed from the focus to the origin of the local frame is $\boldsymbol{R} = (x_1, x_2, H)$, such that $\boldsymbol{n} = \boldsymbol{R}/R$.

[8]B. V. Kostrov, "Seismic moment and energy of earthquakes, and seismic flow of rock", Bull. (Izv.) Acad. Sci. USSR, Earth Physics, **1** 23 (1974) (English translation pp. 13); B. V. Kostrov and S. Das, *Principles of Earthquake Source Mechanics*, Cambridge University Press, NY (1988).

[9]B. F. Apostol, *loc. cit.*

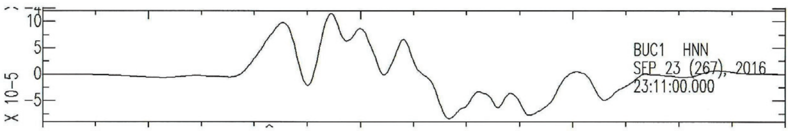

Figure 7.2: P-wave displacement along the South-North direction recorded at Bucharest for the Vrancea earthquake of 23.09.2016.

Unfortunately, the measured longitudinal displacement $\mathbf{v}_l$ is not always along the vector $\mathbf{n}$, which raises a problem of compatibility of the data. This is why we prefer to estimate the depth H of the focus. The experimental determination of the depth of the focus may be affected by errors more than the experimental determination of the epicentral coordinates.

The displacements $\mathbf{v}_l$ and $\mathbf{v}_t$ should be measured from the P- and S-waves of the seismograms for all the three directions, as the maximum value of the displacement (with its sign) on the same temporal side of the seismogram recordings (along the time axis); these recordings have a scissor-like (double-shock) characteristic pattern. The "same temporal side" means either up to the point where these patterns change sign, or away from that point. The "same side" of the P- and S-waves is indicated by arrows in Fig. 7.1. The displacements used here are the envelope of the zoomed out oscillatory curves of the P- and S-waves recorded by seismograms. Instead of the maximum values, mean values may also be used. In addition, the sign of the longitudinal components should be compatible with the position of the focus. For instance, for Vrancea earthquakes recorded in Bucharest, the sign of the longitudinal components should be either $(+,-,+)$ or $(-,+,-)$. We call this constraint the "sign rule". In practice, if the sign rule is not fulfilled, the input data are useless.

We include here an example of reading the displacement from a recording. The raw data are provided by accelerograms. A standard numerical code transforms the accelerations into displacements. The P-wave displacement recorded in Bucharest along the South-North direction for the Vrancea earthquake of 23.09.2016 is shown in Fig. 7.2. The ordinate scale (units 10^{-5}) should be multiplied by 10^3 for a displacement in *cm*. By zooming out this picture, we can see clearly the scissor-like pattern of the displacement. We can read easily the amplitudes on each part of the zero-point; they are $0.14cm$ and $0.8cm$. The mean value of these amplitudes is $0.11cm$. Since our axis points to the South, we must take this value with the negative sign. We assign this value to the left side of the pattern in Fig. 7.2, and preserve this wing for all other seismograms (in order to observe the sign rule).

Let $\mathbf{f} = (f_1,\ f_2,\ f_3)$ be the longitudinal displacement as read from the seismogram and let $\mathbf{g} = (f_1/f,\ f_2/f,\ f_3/f)$. Then, the coordinates of the epicentre should be given by $-Rg_1$, $-Rg_2$, where R is the distance to the focus; they should be as close as possible to the coordinates $x_{1,2}$. Therefore, we minimize the quadratic form $(Rg_1 + x_1)^2 + (Rg_2 + x_2)^2$ and get an estimate

$$R_1 = -\frac{g_1 x_1 + g_2 x_2}{g_1^2 + g_2^2} \tag{7.3}$$

for the focal distance, with a relative error

$$\chi_1 = 1 - \frac{(g_1 x_1 + g_2 x_2)^2}{(g_1^2 + g_2^2)(x_1^2 + x_2^2)} \ . \tag{7.4}$$

Making use of equation (7.3) we get an estimate

$$H_1 = -\sqrt{R_1^2 - (x_1^2 + x_2^2)} \tag{7.5}$$

for the depth of the focus.

Let $\mathbf{v}_t$ be the transverse displacement, measured as described above, and let $\mathbf{t} = \mathbf{v}_t/v_t$. It may happen that $\mathbf{f}$ and $\mathbf{v}_t$ are not perpendicular to each other; we define $\sin\phi = \mathbf{gt}$, where ϕ may be different from zero. This rises again a problem of data compatibility. We define the vector

$$\mathbf{n} = \frac{1}{\cos\phi}(\mathbf{g} - \mathbf{t}\sin\phi) \ \ , \tag{7.6}$$

which is perpendicular to $\mathbf{t}$, and take the longitudinal displacement as

$$\mathbf{v}_l = f\mathbf{n} \ . \tag{7.7}$$

We have now the possibility to get another estimate

$$R_2 = -\frac{n_1x_1 + n_2x_2}{n_1^2 + n_2^2} \tag{7.8}$$

of the focal distance and another estimate

$$H_2 = -\sqrt{R_2^2 - (x_1^2 + x_2^2)} \tag{7.9}$$

of the depth of the focus, with a relative error

$$\chi_2 = 1 - \frac{(n_1x_1 + n_2x_2)^2}{(n_1^2 + n_2^2)(x_1^2 + x_2^2)} \ . \tag{7.10}$$

Finally, we use the mean values $R = (R_1+R_2)/2$ and $H = (H_1+H_2)/2$ for the focal distance and the depth of the focus. In practice, if the angle ϕ is too far from zero, the input data may be discarded, since they lead to large errors.

7.4 Results: earthquake energy and magnitude; focal volume, fault slip

According to the theory,[10] the reduced magnitude of the seismic moment is given by

$$M = (M_{ij}^2/2)^{1/2} = 4\pi\sqrt{2}\rho R^{3/2}\left(c_l v_l^2 + c_t v_t^2\right)^{1/2}\left(c_l^6 v_l^2 + c_t^6 v_t^2\right)^{1/4} \tag{7.11}$$

and the earthquake energy is

$$E = M/2 \tag{7.12}$$

(the magnitude of the seismic moment is $\overline{M} = \sqrt{2}M = (M_{ij}^2)^{1/2}$. Using the Gutenberg-Richter (Hanks-Kanamori) law

$$\lg E = 1.5M_w + 15.65 \tag{7.13}$$

[10]B. F. Apostol, *loc. cit.*

(or $\lg\overline{M} = \frac{3}{2}M_w + 16.1$), we derive the (moment) magnitude of the earthquake

$$M_w = \frac{1}{1.5}\left(\lg E - 15.65\right) \ . \tag{7.14}$$

In these equations R is the focal distance and $\mathbf{v}_l$ is the longitudinal displacement (P-wave, equation (7.7)) as determined above; $\mathbf{v}_t$ is the transverse displacement (S-wave), as measured experimentally; ρ is Earth's mean density (we can take $\rho = 5g/cm^3$) and $c_{l,t}$ are the velocities of the longitudinal and transverse waves (for instance, we may take $c_l = 7km/s$ and $c_t = 3km/s$). All the equations are written in units cm, g, s.

Similarly, the focal volume is given by

$$V = \frac{M}{2\rho c_t^2} \ \ , \tag{7.15}$$

whence we may infer the dimension of the focal region and the magnitude of the fault slip $l = V^{1/3}$.

7.5 Results: seismic-moment tensor; focal strain, focal-activity duration

The seismic-moment tensor is given by

$$M_{ij} = \frac{M}{1-m_4^2}\left[m_i n_j + n_i m_j - m_4\left(m_i m_j + n_i n_j\right)\right] \ \ , \tag{7.16}$$

where

$$m_i = -\frac{c_l^3 v_{li} + c_t^3 v_{ti}}{\left(c_l^6 v_l^2 + c_t^6 v_t^2\right)^{1/2}} \ , \qquad m_4 = -\frac{c_l^3 v_l}{\left(c_l^6 v_l^2 + c_t^6 v_t^2\right)^{1/2}} \tag{7.17}$$

and $\mathbf{n}$ is given by equation (7.6).[11] As discussed above, the components M_{ij} can be viewed as generalized force couples, while the vector $\mathbf{m}$ may be viewed as indicating the direction of a "force" acting in

[11] B. F. Apostol, *loc. cit.*

the focus; m_4 is a measure of the "force" acting along the observation radius (longitudinal "force"). We can check the traceless condition $M_{ii} = 0$ and the covariance condition $m_i^2 = 1$.

The focal strain is given by

$$u_{ij}^0 = \frac{M_{ij}}{2M} \quad , \tag{7.18}$$

where $u_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$ are the strain components for the displacement vector $\mathbf{v}$ (the superscript 0 stands for the focus). The duration of the seismic activity in the focal region[12] is given by

$$T = (2R)^{1/2} \frac{\left(c_l v_l^2 + c_t v_t^2\right)^{1/2}}{\left(c_l^6 v_l^2 + c_t^6 v_t^2\right)^{1/4}} \quad ; \tag{7.19}$$

it is related to the focal volume by

$$V = \frac{4\pi R^2}{c_t^2 T} \left(c_l v_l^2 + c_t v_t^2\right) \quad ; \tag{7.20}$$

hence we may estimate the rate of the focal strain u_{ij}^0/T and the rate of the focal slip l/T (during the seismic activity).

It is useful to have a quick and simple estimation of the order of magnitude of the various quantities introduced here. To this end we use a generic velocity c for the seismic waves and a generic vector $\mathbf{v}$ for the displacement in the far-field seismic waves. From the covariance equation $m^2 = 1$ (equation (7.17)) we get immediately $cT \simeq \sqrt{2Rv}$, which provides an estimate of the duration T of the seismic activity in the focus in terms of the displacement measured at distance R. The focal volume can be estimated as $V \simeq \pi \left(2Rv\right)^{3/2} \simeq \pi(cT)^3$, as expected (dimension l of the focal region of the order cT; the rate of the focal slip is $l/T \simeq c$). The earthquake energy is $E \simeq \mu V \simeq M/2 \simeq 2\rho c^2 V$, where $\mu = \rho c^2$ is the Lame coefficient and M is the reduced magnitude $\left(M_{ij}^2\right)^{1/2} = \sqrt{2}M$ of the seismic moment (and the magnitude of the vector $M_{ij}n_j$). The focal strain is of the order unity, as expected. The magnitude of the earthquake is given immediately by equation (7.14). In addition, we can see the relationship $\lg v =$

[12] B. F. Apostol, *loc. cit.*

$M_w + 10.43 - \lg\left[(2R)(2\pi\rho c^2)^{2/3}\right]$, or $\lg(2Rv) = M_w + 1.83$ (for $\rho = 5g/cm^3$, $c = 5km/s$ and v measured in cm). Hence, we may see that the displacement measured at Bucharest for a Vrancea earthquake of magnitude $M_w = 7$ is of the order $v \simeq 30cm$. Similar estimations can be made for other magnitudes, by using equation (7.14).

7.6 Results: fault geometry

The geometry of the seismic activity in a fault is characterized by the normal $\mathbf{s}$ to the fault (unit vector) and the slip unit vector $\mathbf{a}$ lying on the fault; these two vectors are mutually orthogonal. According to Kostrov representation (and the covariance condition), the seismic-moment tensor is given by

$$M_{ij} = M(s_i a_j + a_i s_j) \ . \tag{7.21}$$

We can see the conditions $M_{ii} = 0$ and $M_{ij}s_i s_j = 0$ (or $M_{ij}a_i a_j = 0$), which, together with the covariance condition $m_i^2 = 1$ (where $m_i = M_{ij}n_j/M$; and $m_4 = M_{ij}n_i n_j/M$), lead to three independent components of the tensor M_{ij}. From equation (7.21) we can see that, apart from the (simultaneous) symmetry operations $\boldsymbol{s} \rightarrow -\boldsymbol{s}$ and $\boldsymbol{a} \rightarrow -\boldsymbol{a}$, which indicate merely a reflection of the fault and the slip, there exists another symmetry, given by $\mathbf{s} \longleftrightarrow \mathbf{a}$, which indicates an important uncertainty. Indeed, any fault slip is accompanied by another fault slip, along an orthogonal direction, as a consequence of matter conservation. It follows that we are not able to make the difference between the direction of the fault and the direction of the slip, because, actually, we have another fault oriented along the slip, and, of course, another slip oriented along the original fault.

The vectors $\mathbf{s}$ and $\mathbf{a}$ are given by[13]

$$\begin{aligned} \mathbf{s} &= \tfrac{\alpha}{\alpha^2-\beta^2}\mathbf{m} - \tfrac{\beta}{\alpha^2-\beta^2}\mathbf{n} \ , \\ \mathbf{a} &= -\tfrac{\beta}{\alpha^2-\beta^2}\mathbf{m} + \tfrac{\alpha}{\alpha^2-\beta^2}\mathbf{n} \ , \end{aligned} \tag{7.22}$$

[13] B. F. Apostol, *loc. cit.*

where

$$\alpha = \sqrt{\frac{1+\sqrt{1-m_4^2}}{2}} \ ,$$
$$\beta = sgn(m_4)\sqrt{\frac{1-\sqrt{1-m_4^2}}{2}} \ ; \tag{7.23}$$

the vector **n** is given by equation (7.6) and the vector **m** and the scalar m_4 are given by equation (7.17). These relations ensure the identity of equation (7.16) with equation (7.21). If we define two orthogonal coordinates $u = a_i x_i$ (along the slip) and $v = s_i x_i$ (along the normal to the fault), then the quadratic form $M_{ij}x_ix_j = const$ defines a hyperbola $uv = const/2M$; its asymptotes are directed along the normal to the fault **s** and the slip in the fault **a**. We call it the seismic hyperbola. For high values of the reduced magnitude M of the seismic moment the seismic hyperbola is tight. Actually, for various *const* in $M_{ij}x_ix_j = const$ we get a hyperboloid directed along the third axis $\mathbf{s} \times \mathbf{a}$.

A similar hyperbola may be derived from equation (7.16) by using the coordinates $\xi = m_i x_i$ (along the vector **m**) and $\eta = n_i x_i$ (along the vector **n**); its equation is $2\xi\eta - m_4(\xi^2 + \eta^2) = const$. We recall that **m** indicates the direction of a "force" acting in the focus; the angle made by the vectors **m** and **n** is given by $\cos\chi = m_4$, the angle made by **n** and **s** (observation radius and the fault direction) is given by $\sin\psi = \sqrt{\left(1 + \sqrt{1 - m_4^2}\right)/2}$ and the angle made by **n** and **a** (observation radius and the fault slip) is $\pi/2 - \psi$.

7.7 Explosions

For explosions, which are isotropic, the moment tensor is a scalar. We write it as $M_{ij} = -M\delta_{ij}$. We have only a longitudinal displacement. The above formulae reduce to

$$M = 2\pi\rho c_l^2(2Rv_l)^{3/2} \ , \ V = \pi(2Rv_l)^{3/2} \ ,$$
$$T = \frac{\sqrt{2Rv_l}}{c_l} \ . \tag{7.24}$$

The "focal" region for explosions is a sphere. The minus sign in the definition of the moment tensor indicates the fact that the slip on a point of the surface of the "focal" sphere is opposite to the direction of the surface element at that point.

Also, we note that both a seismic shear faulting and an explosion produce a longitudinal displacement, such that their distinct contribution cannot be resolved (a superposition of a seismic shear faulting and an isotropic mechanism - the so-called "hybrid" mechanism - cannot be resolved).

7.8 Earthquake of 28.10.2018, Vrancea

We apply here the algorithm described above to two earthquakes.[14] For the Vrancea earthquake of 28.10.2018 the epicentre coordinates are $\theta_E = 45.61°$, $\varphi_E = 26.41°$ and the depth of the focus is $H = -147.8Km$ $(= x_3)$. We use the data from Cernavoda station with coordinates $\theta_0 = 44.3°$, $\varphi_0 = 28.3°$ (coordinates $x_1 = -145.64km$, $x_2 = -125.99km$). The position vector is

$$\boldsymbol{n} = (0.60,\, 0.52,\, 0.61) \ . \tag{7.25}$$

Within the accuracy used here, the vector $\boldsymbol{v}_l$ is directed along the vector $\boldsymbol{n}$, with magnitude $v_l = 0.18cm$, so there is no need to estimate other values of the focus depth and vectors $\boldsymbol{n}$. Noteworthy, the sign rule for Cernavoda is $(+,+,+)$ (or $(-,-,-)$). The vector of the transverse displacement is

$$\boldsymbol{v}_t = (-0.30,\, 0.40,\, -0.08)cm \tag{7.26}$$

(magnitude $v_t = 0.51cm$) and the angle made by $\boldsymbol{v}_l(\boldsymbol{n})$ with $\boldsymbol{v}_t$ is $\simeq 92°$ (which may be viewed as an acceptable departure from orthogonality).

Making use of equations (7.11)-(7.15) we get the energy $E = 4.65 \times 10^{23}erg$, the magnitude of the seismic moment $\overline{M} = 1.30 \times 10^{24}erg$, the magnitude of the earthquake $M_w = 5.33$ and the focal volume $V = 9.6 \times 10^{11}cm^3$. The Institute for Earth's Physics, Magurele,

[14] B. F. Apostol, F. Borleanu and L. C. Cune, "Seismic source and earthquake parameters from local seismic recordings. Earthquakes of 28.10.2018 and 23.05.2016, Vrancea, Romania", Roum. Reps. Phys. **74** 702 (2022).

announced the magnitude $M_w = 5.5$ (www.infp.ro, with a large error $\Delta M_w = 0.7$). We can see that the dimension of the focal volume (the focal slip) is $\simeq 100m$.

Making use of equations (7.16)-(7.20) we get the "force" vector

$$\boldsymbol{m} = (-0.46,\ -0.68,\ -0.56)\ ,\ m_4 = -0.98\ , \tag{7.27}$$

the seismic moment

$$(M_{ij}) = \begin{pmatrix} 1.4 & -7.5 & -1.6 \\ -7.5 & 1.6 & -4.8 \\ -1.6 & -4.8 & -2.8 \end{pmatrix} \times 10^{23} erg \tag{7.28}$$

and the duration of the focal activity $T = 8.7 \times 10^{-3}s$; the focal strain is of the order 10^{-1}, the rate of the focal strain is of the order $10s^{-1}$ and the rate of the focal slip is of the order $10^6 cm/s$. The deviation of M_{ii} from zero in equation (7.28) is a measure of the error of these estimations.

Using equations (7.22) and (7.23), we get the parameters $\alpha = 0.78$, $\beta = -0.63$ and the fault and the slip vectors

$$\begin{aligned} \boldsymbol{s} &= (0.09,\ -0.94,\ -0.26)\ , \\ \boldsymbol{a} &= (0.84,\ -0.09,\ 0.57)\ ; \end{aligned} \tag{7.29}$$

these vectors pierce the Earth's surface at $\theta = 46.05°$, $\varphi = 33.38°$ ($\boldsymbol{s}$) and $\theta = 43.67°$, $\varphi = 26.18°$ ($\boldsymbol{a}$) (see **Appendix**).

7.9 Earthquake of 23.09.2016, Vrancea

The epicentre coordinates for this earthquake are $\theta_E = 45.71°$, $\varphi_E = 26.62°$ and the depth of the focus is $H = -92km$ ($= x_3$). We use the data from Magurele station with coordinates $\theta_0 = 44.35°$, $\varphi_0 = 26.03°$ (coordinates $x_1 = -151.12km$, $x_2 = 45km$). The position vector is

$$\boldsymbol{n} = (0.83,\ -0.25,\ 0.50)\ . \tag{7.30}$$

The direction of the vector $\boldsymbol{v}_l$ is close to the direction of the vector $\boldsymbol{n}$; its magnitude is $v_l = 0.13cm$. The sign rule for Magurele is $(+,-,+)$ (or $(-,+,-)$). The vector of the transverse displacement is

$$\boldsymbol{v}_t = (-0.30,\ -0.30,\ -0.17)cm \tag{7.31}$$

(magnitude $v_t = 0.46cm$) and the angle made by $\boldsymbol{v}_l(\boldsymbol{n})$ with $\boldsymbol{v}_t$ is $\simeq 106°$ (this is a rather large deviation).

Making use of equations (7.11)-(7.15) we get the energy $E = 1.1 \times 10^{23} erg$, the magnitude of the seismic moment $\overline{M} = 3.1 \times 10^{23} erg$, the magnitude of the earthquake $M_w = 4.92$ and the focal volume $V = 2.2 \times 10^{11} cm^3$. The Institute for Earth's Physics, Magurele, announced the magnitude $M_w = 5.5$ (with an error $\Delta M_w = 0.4$). We can see that the dimension of the focal volume (the focal slip) is $\simeq 60m$. The rather large difference in magnitudes arises from the deviation of the vectors $\boldsymbol{v}_{l,t}$ from mutual orthogonality.

Making use of equations (7.16)-(7.20) we get the "force" vector

$$\boldsymbol{m} = (-0.19, -0.57, -0.71) \ , \ m_4 = -0.97 \ , \tag{7.32}$$

the seismic moment

$$(M_{ij}) = \begin{pmatrix} 0.89 & 1.21 & 0.25 \\ 1.21 & 0.23 & 1.45 \\ 0.25 & 1.45 & -0.8 \end{pmatrix} \times 10^{23} erg \tag{7.33}$$

and the duration of the focal activity $T = 6 \times 10^{-3} s$; the focal strain is of the order 10^{-1}, the rate of the focal strain is of the order $10 s^{-1}$ and the rate of the focal slip is of the order $10^6 cm/s$. The deviation of M_{ii} from zero in equation (7.33) is a measure of the error of these estimations.

Using equations (7.22) and (7.23), we get the parameters $\alpha = 0.79$, $\beta = -0.61$ and the fault and the slip vectors

$$\begin{gathered} \boldsymbol{s} = (0.29, 0.77, -0.21) \ , \\ \boldsymbol{a} = (0.69, 0.07, 0.88) \ ; \end{gathered} \tag{7.34}$$

these vectors pierce Earth's surface at $\theta = 45.88°$, $\varphi = 22.24°$ ($\boldsymbol{s}$) and $\theta = 44.1°$, $\varphi = 26.7°$ ($\boldsymbol{a}$).

7.10 Concluding remarks

The practical application of the theory of determining the seismic source and the earthquake parameters from local seismic recordings of the P and S seismic waves[15] is presented, with a detailed emphasis

[15] B. F. Apostol, *loc. cit.*

on the specific points of its implementation. Special attention is given to the input parameters, as read from seismograms; these parameters, which are the amplitudes of the ground displacement, should satisfy certain compatibility conditions. The optimization procedure described above can be employed to improve compatibility. It is shown how the earthquake energy and magnitude can be derived, as well as the volume of the focal region and the focal slip. Also, it is shown how to determine the tensor of the seismic moment, the focal strain and the duration of the seismic activity in the earthquake focus. We describe how to deduce the orientation of the fault and the direction of the focal slip. The particular case of an isotropic explosion is also presented. The procedure is applied to two Vrancea earthquakes (28 October, 2018, and 23 September, 2016), both with magnitude $M_w = 5.5$. The errors implied by the practical application of this theory are discussed. Particularly interesting is a rapid estimation, by hand, of the earthquake parameters, which is described above. The procedure presented here can be implemented by means of a numerical computing program with modest resources, and the results can be obtained in real time. The procedure is routinely applied at the Institute of Earth's Physics in Magurele, Romania.

Task #6 of Practical Seismology is the implementation of the procedure described above, by a numerical code, for determining the seismic moment, the magnitude, the energy and the source parameters of the Vrancea earthquakes in real time, from readings of the seismograms of the recorded P and S seismic waves.

7.11 Appendix

It may be of interest to determine the points where the vectors **s** and **a** pierce Earth's surface. For this it is necessary to express all the vectors in the reference frame of the Earth (a sphere). We have the vector which determines the origin of the local frame, the vector which determines the focus and the vector **s** (or **a**) with the origin in the focus. The point of interest on Earth's surface corresponds to a vector $\lambda\mathbf{s}$ (or $\lambda\mathbf{a}$), where λ has a well-determined value. We express this vector in Earth's frame and requires it to be on Earth's surface;

this condition leads to the equation

$$\lambda^2 + 2\lambda \left[R_0 s_3 - R(\mathbf{ns})\right] - 2R_0 H = 0 \ ; \tag{7.35}$$

we need to choose for λ the smallest absolute value of the roots.

A simplified version of these calculations can be done for points close to the local observation point and the epicentre, such that we may approximate the Earth's surface by a plane surface. The corresponding equations are

$$\begin{aligned} H\tfrac{s_1}{s_3} + x_1 &= -R_0\theta \ , \\ H\tfrac{s_2}{s_3} + x_2 &= R_0 \cos\theta^{'} \cdot \varphi \end{aligned} \tag{7.36}$$

($s_3 > 0$); the coordinates of the intersection point are $\theta^{'} = \theta_0 + \theta$ and $\varphi^{'} = \varphi_0 + \varphi$.

8 Quasi-static Deformations

8.1 Introduction

We have described in this book the derivation of the seismic moment and the parameters of the seismic source from measurement of the displacement caused by the P and S seismic waves on Earth's surface. We show below that the same problem can be solved by measuring the static displacement caused by a seismic source in the epicentral zone. It is admitted that the continuous accumulation of the tectonic stress may be gradually discharged, to some extent and with intermittence, causing quasi-static crustal deformations of Earth's surface in seismogenic zones.[1] We show here that measurements of these deformations may give, besides qualitative information about the seismic activity, the depth of the focus and the focal volume, the opportunity of determining the tensor of the seismic moment for a shear faulting.[2] We use here the displacement derived previously for a homogeneous isotropic half-space with a free surface and tensor point forces generated by a

[1]F. C. Frank, "Deduction of earth strains from survey data", Bull. Seism. Soc. Am. **56** 35 (1966); J. C. Savage and R. O. Burford, "Geodetic determination of relative plate motion in central California", J. Geophys. Res. **78** 832 (1973); J. C. Savage, "Strain accumulation in western United States", Ann. Rev. Earth Planet. Sci. **11** 11 (1983); K. L. Feigl, D. C. Agnew, Y. Bock, D. Dong, A. Donnellan, B. H. Hager, T. A. Herring, D. D. Jackson, T. H. Jordan, R. W. King, S. Larsen, K. M. Larson, M. M. Murray, Z. Shen and F. W. Webb, "Space geodetic measurement of crustal deformation in central and southern California, 1984-1992", J. Geophys. Res. **98** 21677 (1993); S. N. Ward, "A multidisciplinary approach to seismic hazard in southern California", Bull. Seism. Soc. Am. **84** 1293 (1994); Working Group on California Earthquake Probabilities, "Seismic hazards in southern California: probable earthquakes, 1994-2024", Bull. Seism. Soc. Am. **85** 379 (1995); J. C. Savage and R. W. Simpson, "Surface strain accumulation and the seismic moment tensor", Bull. Seism. Soc. Am. **87** 1345 (1997).

[2]B. F. Apostol, *Seismology*, Nova, NY (2020).

seismic moment in a focus localized inside the half-space.[3]

8.2 Surface displacement

The static deformations produced by a tensor point force $\mathbf{f}$ in a homogeneous isotropic elastic half-space are given by the equation of elastic equilibrium

$$\Delta\mathbf{u}+\frac{1}{1-2\sigma}\mathrm{grad}\,\mathrm{div}\mathbf{u}=-\frac{2(1+\sigma)}{E}\mathbf{f}\ \ , \tag{8.1}$$

where $\mathbf{u}$ is the displacement vector (with components u_i, $i=1,2,3$), E is the Young modulus and σ is the Poisson ratio. The components of the force density are given by

$$f_i=M_{ij}\partial_j\delta(\mathbf{r}-\mathbf{r}_0)\ \ , \tag{8.2}$$

where $\mathbf{r}_0$ is the position of the focus and M_{ij} is the tensor of the seismic moment. It is convenient to write $\overline{\mathbf{f}}=-\left[2(1+\sigma)/E\right]\mathbf{f}$ and $\overline{M}_{ij}=-\left[2(1+\sigma)/E\right]M_{ij}$ (reduced force and seismic moment). Equation (8.1) is solved for a half-space $z<0$, with free surface $z=0$, the position of the focus being $\mathbf{r}_0=(0,\,0,\,z_0)$, $z_0<0$ (epicentral frame); we use the radial coordinate $\rho=(x^2+y^2)^{1/2}$ for the in-plane coordinates x, y and $x_1=x$, $x_2=y$, $x_3=z$. We use the labels α, β, γ, etc for the components $1,2$.

The components u_α given by equations (8.1) are vanishing for $\rho\longrightarrow 0$ and go like $1/\rho^2$ for $\rho\longrightarrow\infty$; they have a maximum value for ρ of the order $\mid z_0\mid$. The component u_3 goes like $1/z_0^2$ for $\rho\longrightarrow 0$ and $1/\rho^2$ for $\rho\longrightarrow\infty$.[4] It is convenient to give these displacement components for ρ close to zero, *i.e.* in the seismogenic zone (close to a presumable

[3]B. F. Apostol, "Elastic displacement in a half-space under the action of a tensor force. General solution for the half-space with point forces", J. Elast. **126** 231 (2017); *loc. cit.*

[4]B. F. Apostol, *loc. cit.*

epicentre). We get

$$u_\alpha = \tfrac{1}{16\pi}\left[4(1-2\sigma)\overline{M}_{33} - (3+2\sigma)\overline{M}_0\right]\tfrac{x_\alpha}{|z_0|^3} +$$

$$+\tfrac{1}{8\pi}(1-2\sigma)\tfrac{\overline{M}_{\alpha\beta}x_\beta}{|z_0|^3} + ... \ , \tag{8.3}$$

$$u_3 = \tfrac{1}{8\pi z_0^2}\left[2(3-2\sigma)\overline{M}_{33} - (1+2\sigma)\overline{M}_0\right] +$$

$$+\tfrac{\overline{M}_{3\alpha}x_\alpha}{2\pi|z_0|^3} + ... \ ,$$

where $\overline{M}_0 = \overline{M}_{ii}$ is the trace of the tensor $\overline{M}_{ij}$.
A simplified numerical estimation of the unknowns (components of the seismic moment) can be obtained as follows. We assume $M_0 = 0$ (as for a shear faulting), replace all the components of the seismic-moment tensor in equations (8.3) by a mean value $\overline{M}$ and average over the orientation of the vector $\boldsymbol{\rho}$; we denote the resulting u_3 by u_v (vertical component) and introduce u_h (horizontal component) by $u_h = \left(u_1^2 + u_2^2\right)^{1/2}$; we get approximately

$$u_h \simeq \frac{(1-2\sigma)\left|\overline{M}\right|}{4\pi}\frac{\rho}{\mid z_0 \mid^3} \ , \ u_v \simeq \frac{(3-2\sigma)\overline{M}}{4\pi z_0^2} \ ; \tag{8.4}$$

hence, we get immediately the depth of the focus

$$\mid z_0 \mid \simeq \frac{1-2\sigma}{3-2\sigma}\left|u_v\right|/(\partial u_h/\partial\rho) \tag{8.5}$$

and the mean value $\overline{M} = 4\pi z_0^2 u_v/(3-2\sigma)$ of the (reduced) seismic moment. Making use of $\overline{M}_{ij} = -\left[2(1+\sigma)/E\right]M_{ij}$ we have

$$M_{av} \simeq -\frac{2\pi E}{(1+\sigma)(3-2\sigma)}z_0^2 u_v \tag{8.6}$$

for the mean value M_{av} of the seismic moment M_{ij}. Since the small displacement values u_h, u_v are affected by errors, the determination of the mean value of the seismic moment may be viewed as satisfactory. For $M_{av} = 10^{22} dyn \cdot cm$ (which would correspond to an earthquake with magnitude $M_w = 4$ by the Gutenberg-Richter law $\lg M_{av} =$

$1.5M_w + 16.1$), Young modulus $E = 10^{11} dyn/cm^2$, $\sigma = 0.25$ and depth $\mid z_0 \mid = 100km$ we get a vertical displacement $u_v \simeq 1\mu m$; we can see that the static surface displacement is indeed small.

A rough estimate for the elastic energy stored by the static deformation is given by $\mathcal{E} \simeq 4\pi z_0^2 E \mid u_v \mid \simeq 2(1+\sigma)(3-2\sigma)\,|M_{av}|$; it is also given by $\mathcal{E} \simeq \mu V$, where μ is the Lame coefficient and V is the focal volume ($\mu = E/2(1+\sigma)$; the other Lame coefficient is $\lambda = E\sigma/(1-2\sigma)(1+\sigma)$); making use of the approximations introduced above, we get $V \simeq 8\pi(1+\sigma)z_0^2 \mid u_v \mid$. For $\mid z_0 \mid = 100km$ and $u_v = 1\mu m$ ($\sigma = 0.25$) we get a volume $V \simeq 10^5 \pi m$, *i.e.* a linear dimension $l \simeq 500m$. Similarly, from equations (8.3) we get an estimate $u_{ij} \sim V / \mid z_0 \mid^3$ for the surface strain; using the numerical data above, it is of order 1Å.[5]

8.3 General form

Making use of the general results of static deformations,[6] the displacement components given by equation (8.3) can be written in a general form (for $M_0 = 0$) as

$$u_i = \{[2(3-2\sigma)\overline{M}_4^{(n)} - (9-10\sigma)\overline{M}_4^{(nv)}]n_i - -4\overline{M}_4^{(n)} v_i + (1-2\sigma)\overline{M}_{ij} v_j\}\frac{1}{8\pi z_0^2} \quad , \tag{8.7}$$

where

$$\mathbf{n} = (x_\alpha, z - z_0)/\mid z_0 \mid \ , \ \ \mathbf{v} = (x_\alpha, z)/\mid z_0 \mid \ , \qquad \overline{M}_4^{(n)} = \overline{M}_{ij} n_i n_j \ , \ \ \overline{M}_4^{(nv)} = \overline{M}_{ij} n_i v_j \ ; \tag{8.8}$$

in equations (8.7) and (8.8) we retain only contributions linear in x_α and in the limit $z \to 0$. Within these restrictions the form given by equation (8.7) is unique. In these equations

$$\overline{M}_i = \overline{M}_{ij} v_j \simeq \frac{\overline{M}_{i\alpha} x_\alpha}{\mid z_0 \mid} \tag{8.9}$$

[5] A static deformation may diffuse, such that the corresponding focal volume is larger than the focal volume of a sudden earthquake discharge.

[6] B. F. Apostol, *loc. cit.*

are the components of a vector and

$$\overline{M}_4^{(n)} \simeq 2\overline{M}_3 + \overline{M}_{33} \ , \ \overline{M}_4^{(nv)} \simeq \overline{M}_3 \tag{8.10}$$

are scalars. Taking the scalar product $\mathbf{nu} \simeq u_3$ in equation (8.7), we get

$$\overline{M}_4^{(n)} = \frac{4\pi z_0^2 u_3 + 4(1-\sigma)M_3}{3-2\sigma} \ ; \tag{8.11}$$

inserting this $\overline{M}_4^{(n)}$ and $\overline{M}_4^{(nv)} \simeq \overline{M}_3$ in equation (8.7) we get

$$u_\alpha = \frac{1-2\sigma}{3-2\sigma}\frac{x_\alpha}{\mid z_0 \mid}u_3 + \frac{1-2\sigma}{8\pi z_0^2}\overline{M}_\alpha \tag{8.12}$$

(and the identity $u_3 = u_3$). This equation gives

$$\overline{M}_\alpha = 8\pi z_0^2 \left(\frac{1}{1-2\sigma}u_\alpha - \frac{1}{3-2\sigma}\frac{x_\alpha}{\mid z_0 \mid}u_3 \right) \tag{8.13}$$

(and $M_\alpha = -\left[E/2(1+\sigma)\right]\overline{M}_\alpha$) as functions of the measured quantities u_α, u_3 and x_α; $\overline{M}_4^{(nv)}$ and $\overline{M}_4^{(n)}$ are given by equations (8.10) and (8.11) as functions of u_3 and the parameter $\overline{M}_3$. This is the maximal information provided by measuring the static displacement in a seismogen zone; the parameter z_0 remains undetermined; we can use its numerical estimation given above (equation (8.5)).

8.4 Seismic moment

We assume that the components M_α of the vector $\mathbf{M}$ are determined from data, according to equation (8.13); the component M_3 will be determined shortly. The scalars $M_4^{(nv)} \simeq M_3$ and $M_4^{(n)}$ are given by equations (8.10) and (8.11), respectively; they depend on the parameter M_3. Parameters z_0 (focus depth) and the focal volume V remain undetermined. The order-of-magnitude estimations given above (equation (8.5) and below) may be used for them.[7]

[7]B. F. Apostol, "Near-field seismic motion", J. Theor. Phys. **328** (2021).

In order to determine the seismic moment we use its expression derived before for a shear faulting; it is given by

$$M_{ij} = M^0(s_i a_j + s_j a_i) \ , \ i, j = 1, 2, 3 \ , \tag{8.14}$$

where $M^0 = 2\mu V$ and s_i, a_i are the components of two orthogonal unit vectors **s** and **a**: **s** is normal to the fault plane and **a** is directed along the fault displacement (fault sliding). We can see that equation (8.8) implies $M_0 = M_{ii} = 0$. We assume that the measured data of the static displacement satisfy this condition. In addition, we assume that M^0 is a known parameter.

We introduce the scalar products $A = \mathbf{av}$ and $B = \mathbf{sv}$ and write

$$A\mathbf{s} + B\mathbf{a} = \mathbf{m} \ , \ B\mathbf{s} + A\mathbf{a} = \mathbf{v} \tag{8.15}$$

from equation (8.14), where $\mathbf{m} = \mathbf{M}/M^0$; we solve this system of equations for **s** and **a** with the conditions $s^2 = a^2 = 1$, $\mathbf{sa} = 0$. We note that equation (8.14) is invariant under the symmetry operations $\mathbf{s} \longleftrightarrow \mathbf{a}$ and $\mathbf{s}, \mathbf{a} \longleftrightarrow -\mathbf{s}, -\mathbf{a}$ (and $\mathbf{s} \longleftrightarrow -\mathbf{a}$); consequently, it is sufficient to retain one solution of the system of equations (8.15) (it has multiple solutions), all the other being given by these symmetry operations. We get

$$\mathbf{s} = \tfrac{A}{A^2-B^2}\mathbf{m} - \tfrac{B}{A^2-B^2}\mathbf{v} \ , \ \mathbf{a} = -\tfrac{B}{A^2-B^2}\mathbf{m} + \tfrac{A}{A^2-B^2}\mathbf{v} \tag{8.16}$$

and

$$A^2 + B^2 = m^2 = v^2 \ , \ 2AB = v^2 m_4 \ , \tag{8.17}$$

where $m_4 = \mathbf{mv}/v^2 = M_{ij}v_i v_j / v^2 M^0$. From $m^2 = v^2$ we get the component M_3 as given by

$$M_3^2 = \left(M^0\right)^2 v^2 - M_\alpha^2 \ ; \tag{8.18}$$

we may take

$$A = v\sqrt{\frac{1+\sqrt{1-m_4^2}}{2}} \ , \ B = sgn(m_4) \cdot v\sqrt{\frac{1-\sqrt{1-m_4^2}}{2}} \tag{8.19}$$

as a solution of the system of equations (8.17); this solves the problem of determining the seismic moment from the measurements of the surface static displacement. From equation (8.14) the seismic-moment tensor is given by

$$M_{ij} = \frac{M^0}{v^2(1-m_4^2)}\left[m_i v_j + m_j v_i - m_4\left(m_i m_j + v_i v_j\right)\right] \ ; \qquad (8.20)$$

the vector $\mathbf{v}$ is known from equation (8.8) ($z \to 0$, $v = \rho / \mid z_0 \mid$) and the vector $\mathbf{m}$ is known from equations (8.13) and (8.18) (with z_0 and M^0 as known parameters); the scalar m_4 is given by $m_4 = M_\alpha v_\alpha / v^2 M^0$. The component M_3 does not enter the expression of m_4; it is included in M_{ij}. The quadratic form $M_{ij} x_i x_j = const$ is a hyperbola; its asymptotes indicate the fault plane (vector $\mathbf{s}$) and the fault slip (vector $\mathbf{a}$).

The isotropic case $M_{ij} = -M^{is}\delta_{ij}$, where $M^{is} = 2(2\mu + \lambda)V$, implies a surface displacement

$$\mathbf{u} = \frac{M^{is}(1+\sigma)}{4\pi z_0^2 E}\left[(3-10\sigma)\mathbf{n} - (3-\sigma)\mathbf{v}\right] \ \ , \qquad (8.21)$$

the vector $\mathbf{M}$ being given by $\mathbf{M} = -M^{is}\mathbf{v}$. The energy can be estimated as $\mathcal{E} = M^{is}/2 = 4\pi z_0^2 E \mid u_v \mid$,which leads to a focal volume $V = [4\pi(1+\sigma)(1-2\sigma)/(1-\sigma)] z_0^2 \mid u_v \mid$.

Task #7 of Practical Seismology is to deduce the average seismic moment M_{av}, the depth z_0 and the volume of the focal region V by using static deformations measured at the surface of the Earth in the epicentral zone (equations (8.5) and (8.6)). Also, the tensor of the seismic moment can be derived from these static displacements (equation (8.20)).

9 Structural Engineering

9.1 Embedded bar

The response of various inhomogeneities to seismic movements is a subject of utmost importance for the design of buildings (structural engineering); on the other hand, local amplification is reported in soil displacement, velocities and acceleration during an earthquake, which is an indication of internal inhomogeneities in the Earth's crust.[1] The most convenient model for the response of an inhomogeneity to an earthquake movement is the embedded vibrating bar.[2]

Let us assume that a vertical elastic bar with uniform cross-section is fixed in the ground at one end, having a length l above the ground surface; the bar end above the ground is free. Under the action of the seismic waves the buried end of the bar is set in motion. We assume the cross-sectional dimensions of the bar being much smaller than the bar length, so we may limit ourselves only to the z-dependence of the displacement, where z is the vertical coordinate (along the bar). At the same time, we consider the length of the bar and the excitation sufficiently small, such that the bar does not enter the regime of flexural elasticity (bending).

The elastic motion of the bar implies transverse displacements and a longitudinal displacement, each with its own wave velocity. The general equation of motion of the local displacement reads

[1] P. Y. Bard and M. Bouchon, "The two dimensional resonance of sediment filled valleys", Bull. Seism. Soc. Am. **75** 519 (1985); J. F. Semblat, M. Kham, E. Parara, P. Y. Bard, K. Pitilakis, K. Makra and D. Raptakis, Site effects: basin geometry vs soil layering, Soil Dyn. Earthq. Eng. **25** 529 (2005); J. F. Semblat and A. Pecker, Waves and vibrations in soils: earthquakes, traffic, shocks, construction works, IUSS Press, Pavia (2009).

[2] S. P. Timoshenko, "On the transverse vibrations of bars of uniform cross-section", Phil. Mag. **43** 125 (1922).

$$\ddot{u} - c^2 \frac{\partial^2 u}{\partial z^2} = 0 \ , \tag{9.1}$$

where u is the displacement and c denotes the elastic wave velocity. This equation is solved with the boundary conditions of a free end and a given displacement at the fixed end, *i.e.*

$$\tfrac{\partial u}{\partial z} \mid_{z=l}= 0 \ , \ u \mid_{z=0}= u_0(t) \ . \tag{9.2}$$

We consider seismic excitations which have a general aspect of shocks, *i.e.* they are concentrated at the initial moment of time. This is valid for both the primary P and S waves, as well as for the main shock produced by the so-called surface waves. Consequently, we assume first a shock-like ground motion $u_0(t) = Tu_0\delta(t)$, where T is a measure for the duration of the shock. Apart from the original shock which propagates along the bar, the local displacement in the bar is also a superposition of eigenfrequencies of the bar, given by

$$u(z,t) = \frac{1}{2} u_0 T\delta(t - z/c) + u_0 \frac{cT}{l} \sum_n \sin\omega_n t \cdot \sin\omega_n z/c \ , \tag{9.3}$$

where $\omega_n = (2n+1)\pi c/2l$, $n = 0, 1, 2, ...$ (the roots of the equation $\cos\omega_n l/c = 0$). We can see that an amplification factor $g = cT/l$ appears. From equation (9.3), similar amplification factors appear for velocities and accelerations ($g\omega_n$, $g\omega_n^2$, respectively). Typical values of the velocity of the elastic waves in the bar are $c \simeq 3 \times 10^3 m/s$; for a short duration $T = 0.1s$ we get $g = 10$ for a length $l = 30m$. We can see that the displacement, velocity and acceleration amplitudes in the bar could be enhanced in comparison with their ground counterparts. This is why we may call the parameter g the amplification factor.[3]

However, a pulse with a finite duration T excites mainly frequencies ω_n up to $\simeq \pi/T$. Therefore, the amplification parameter is subject to the condition

$$\omega_n T = \frac{(2n+1)\pi}{2} g \leq \pi \ , \tag{9.4}$$

[3] B. F. Apostol, *The Theory of Earthquakes*, Cambridge International Science Publishers, Cambridge (2017); *Introduction to the Theory of Earthquakes*, Cambridge International Science Publishers, Cambridge (2017).

which implies values for g as high as the order of unity, corresponding to the fundamental frequency $\omega_0 = \pi c/2l$ ($n = 0$). In addition, it is well known that the seismic spectrum includes a range of frequencies extending up to $\simeq 10s^{-1}$, which is far below a fundamental frequency of the order $c/l \simeq 100s^{-1}$ for $c \simeq 3 \times 10^3 m/s$ and $l = 30m$. Therefore, it is unlikely that a short pulse can excite normal modes which might lead to appreciable amplification factors in reasonable conditions. However, the situation is different if the pulse includes resonance frequencies.

Also, we note that if the pulse is applied at some point on the bar, different from the bar ends, then we deal in fact with two bars. The boundary conditions are the continuity of the displacement at the point of application of the excitation, the equality of the displacement with the excitation at that point and the conditions at the two ends.

9.2 Oscillating shock

If the ground displacement has the form of a harmonic oscillation $u_0(t) = u_0 \cos\omega_0 t$, the displacement in the bar is[4]

$$u(z,t) = u_0 \cos\omega_0 t \frac{\cos\omega_0(z-l)/c}{\cos\omega_0 l/c} \ ; \tag{9.5}$$

if ω_0 happens to be an eigenfrequency of the bar ($\omega_0 = \omega_n$), then the amplitude increases indefinitely, and the dangerous resonance phenomenon occurs (we recall that the eigenfrequencies ω_n are the roots of the equation $\cos\omega_n l/c = 0$).

Let us assume a ground motion given by

$$u_0(t) = u_0 \theta(t) e^{-\alpha t} \cos\omega_0 t \ , \tag{9.6}$$

where $\theta(t) = 1$ for $t > 0$, $\theta(t) = 0$ for $t < 0$ is the step function and $0 < \alpha \ll \omega_0$; it represents an oscillating shock with a sharp wavefront, attenuated in time with the rate α, which is deemed to model the seismic main shock with its long tail (produced by the so-called surface waves). We may leave aside the original propagating

[4]B. F. Apostol, *loc. cit.*

shock. We can compute the response of the bar for $\omega \neq \omega_n$.[5] At resonace

$$u(z,t) = u_0 \frac{c}{l} \frac{1-e^{-\alpha t}}{\alpha} \sin\omega_0 t \cdot \sin\omega_0 z/c \ . \tag{9.7}$$

We can see that the displacement amplitudes at resonance are $(c/l\alpha)u_0$, *i.e.* in the amplification factor $g = cT/l$ the duration T is replaced by $1/\alpha$, as expected. We note that for $\omega_0 = 0$ the amplitude is reduced to $(c/l\omega_n)u_0$. Similarly, the response velocity and acceleration include factors $u_0\omega_0$ and $u_0\omega_0^2$, respectively, which now can be viewed as corresponding to the ground velocity and acceleration; the amplification factor for these quantities is $g = c/l\alpha$, as for the displacement. It is worth emphasizing that amplification factors of the type $g = c/l\alpha$ may attain high values.

Task #8 of Practical Seismology is to check displacements (velocities and accelerations) given by equations (9.3) and (9.7) against the values measured by sensors in high buildings, in order to derive useful parameters for an informed design, maintenance, safety measures, etc.

9.3 Buried bar. Site amplification factors

We consider now a bar completely buried in the ground, with both ends free (its orientation is immaterial); we assume that the bar moves freely in the ground, its displacement being superposed over the displacement u_0 of the ground. We assume a ground excitation

$$u_0(z,t) = u_0\theta(t)e^{-\alpha t}\cos\omega_0 t \cdot \cos\kappa_0 z \ , \tag{9.8}$$

where $\kappa_0 = \omega_0/c_0$, c_0 being the wave velocity in the soil. The eigenfrequencies of the bar are $\omega_n = n\pi c/l$, for $n = 1, 2, 3, ...$(the roots of the equation $\sin\omega_n l/c = 0$). We can compute the response of this bar to the soil excitation for $\omega_0 \neq \omega_n$(non-resonance). At resonance $(\omega_0 = \omega_n)$ the displacement along the bar is

$$u(z,t) = (-1)^n u_0 \frac{c^2}{c_0 l} \sin\omega_0 l/c_0 \frac{1-e^{-\alpha t}}{\alpha} \sin\omega_0 t \cdot \cos\omega_0 z/c \ . \tag{9.9}$$

[5] B. F. Apostol, *loc. cit.*

We can see the occurrence of an amplification factor

$$g = \frac{c^2}{c_0 l \alpha} \sin \omega_0 l / c_0 \ , \tag{9.10}$$

which now has a more complex structure; it depends on the wave velocity c_0 and the frequency ω_0 of the excitation (it is a spectral amplification factor).

This result may throw an interesting light upon the so-called amplification site effect. It is well known that the ground displacement, velocity and acceleration may exhibit large local variations from site to site. In the light of the above result it is easy to see that a local inhomogeneity surrounded by a different environment may behave as a buried bar, and the normal modes set in this inhomogeneity may exhibit large amplifications factors. The effect is enhanced for a stiff inhomogeneity ($c \gg c_0$), but for low attenuation factors α it may appear also for soft inhomogeneities. The conditions for its occurrence are the resonance and seismic waves with wavelengths shorter than the linear dimension of the inhomogeneity. We note that it is easy to see that the bar-shape of the inhomogeneity is irrelevant; the amplification may occur for inhomogeneities of any shape; the necessary conditions are $c_0/\omega_0 < l$ (excitation wavelength shorter than the dimension of the inhomogeneity) and $\omega_0 = \omega_n = c\alpha_n/l$, where α_n is a numerical coefficient which gives the eigenfrequency ω_n (increasing with increasing n); these conditions imply $c_0 < c\alpha_n$.

We can see that there exists a discontinuity between the soil displacement u_0 and the displacement u of the bar at the points of the bar. If we allow for a finite extension d of the bar, along, say, the transverse direction x, then this discontinuity disappears. If the inhomogeneity is gradually disappearing along the longitudinal direction, the discontinuity disappears along this direction too. If the inhomogeneity has sharp ends with respect to the surrounding medium, the discontinuity remains.

Task #9 of Practical Seismology is to assess the dimension and the nature of buried inhomogeneities from recorded amplification factors, according to equation (9.10).

9.4 Coupled harmonic oscillators

It is usual to view a building-foundation structure as two coupled harmonic oscillators, each with its own mass and eigenfrequency. According to the solution to this problem,[6] the coupling reduces the low eigenfrequency and raises the high eigenfrequency.

For a realistic use of the coupled-oscillator model we consider the two oscillators as corresponding to a building (oscillator 2) and its foundation (oscillator 1). For a stiff foundation (higher eigenfrequency) the eigenfrequencies of the building are reduced to an appreciable extent (down to zero), while the eigenfrequencies of the foundation are increased by the coupling. For a soft foundation (lower eigenfrequency) the situation is reversed, the eigenfrequencies of the building are raised by the coupling and those of the foundation are reduced.

If the excitation is at resonance with the building, the original damped excitation is lost in time and for long time both the building and the foundation oscillate with the resonance frequency of the building; the amplitudes of the oscillations are enhanced by the attenuation factor of the excitation, as expected; the oscillation amplitude of the foundation is controlled by the exciting force, while the amplitude of the building is controlled by the coupling constant. We note that we have considered oscillations without a damping factor; a damping factor affects the contribution of the normal modes and adds to the attenuation factor of the excitation.

As regards a possible seismic base isolation of the buildings by designing special foundations we may say that the answer is not definite. The coupling of the two structures reduces the low frequency (either of the building or of the foundation) and raises the higher frequency, without definitely removing the building frequency from the range of the seismic shock; at resonance, especially for long-lasting shocks, the amplification factors may attain appreciable values.

[6] B. F. Apostol, *loc. cit.*

9.5 Coupled bars

The model of coupled vibrating bars is useful for accounting for voids in buildings.[7] Let us assume a bar with length l fixed at $z = 0$ to another long bar with length l_0; we denote the former bar by 1 and the latter bar by 2. The equations of elastic motion in the two bars are

$$\ddot{u}_1 - c_1^2 u_1^{''} = 0 \ , \ \ddot{u}_2 - c_2^2 u_2^{''} = 0 \ , \tag{9.11}$$

where $u_{1,2}$ are the displacements in the two bars; the boundary conditions are

$$\begin{gathered} u_2 \mid_{z=-l_0} = u_0(t) \ , \ u_1 \mid_{z=0} = u_2 \mid_{z=0} \ , \\ \mu_1 u_1^{'} \mid_{z=0} = \mu_2 u_2^{'} \mid_{z=0} \ , \ u_1^{'} \mid_{z=l} = 0 \ , \end{gathered} \tag{9.12}$$

which signify a ground motion applied to the lower end $z = -l_0$ of bar 2, the continuity of the displacement at the joining point (the bars are rigidly connected to each other), the absence of the force at the interface $z = 0$ and the free upper end $z = l$; the force is written for a shear displacement; for compression (dilatation) the rigidity moduli $\mu_{1,2}$ should be replaced by $\lambda_{1,2} + 2\mu_{1,2}$. The solutions of these equations show that the eigenfrequencies are controlled by the elastic properties of the "softer" bar. This result gives an indication regarding the vibration properties of bars with a composite structure (*e.g.*, including voids).

[7] B. F. Apostol, *loc. cit.*

10 Spectral (Site) Response

10.1 Seismic displacement

A typical seismogram, recorded at Earth's surface, consists of a succession of P and S seismic waves, followed by a main shock. The displacement of the P and S waves is given by

$$\begin{aligned} \boldsymbol{u}_P &= -\tfrac{TM_4}{4\pi\rho c_l^3 R}\boldsymbol{n}\delta'(t-R/c_l) \ , \\ \boldsymbol{u}_S &= -\tfrac{T(M_4\boldsymbol{n}-\boldsymbol{M})}{4\pi\rho c_t^3 R}\delta'(t-R/c_t) \ , \end{aligned} \tag{10.1}$$

where T is the duration of the seismic activity in the focus, $\boldsymbol{n}$ is the unit vector from the focus to the observation point placed at distance R from the focus, ρ is the density of the homogeneous and isotropic elastic body (the Earth), $c_{l,t}$ are the propagation velocities of the longitudinal and transverse elastic waves and $M_i = M_{ij}n_j$, $M_4 = M_{ij}n_i n_j$, where M_{ij} is the tensor of the seismic moment. $\boldsymbol{M}$ and M_4 can be called the seismic moment vector and the seismic moment scalar. These formulae are valid for a focus localized in time (short duration T) and space (small length l), where a seismic tensorial force acts. Both T and l are much smaller than the times and distances of interest. The seismic waves given by equation (10.1) are spherical-shell waves. The P wave is longitudinal, while the S wave is transverse.[1]

The parameters T and l are independent. Accordingly, the function $\delta'(t-R/c_{l,t})$ in equations (10.1) is viewed as a function localized on the observation point P, placed at distance R from the seismic focus F, with a temporal width $\Delta t = l/c_{l,t}$ ($\ll t$) and a function localized at a fixed moment of time with a spatial width $l_{l,t} = c_{l,t}T$ ($\ll R$). For

[1]B. F. Apostol, "Elastic waves inside and on the surface of a half-space", Quart. J. Mech. Appl. Math. **70** 289 (2017).

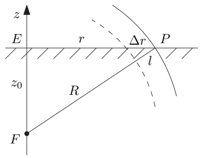

Figure 10.1: Seismic spot on Earth's surface (focus F, epicentre E, observation point P).

a point P placed on Earth's plane surface at distance $r_{l,t}$ from the epicentre E we have

$$R_{l,t}^2 = r_{l,t}^2 + z_0^2 \ , \ (R_{l,t} + l_{l,t})^2 = (r_{l,t} + \Delta r_{l,t})^2 + z_0^2 \ , \qquad (10.2)$$

where z_0 is the depth of the focus and

$$\Delta r_{l,t} = \frac{2R_{l,t}l_{l,t} + l_{l,t}^2}{r_{l,t} + \sqrt{r_{l,t}^2 + 2R_{l,t}l_{l,t} + l_{l,t}^2}} \qquad (10.3)$$

is the spread of the seismic spot on Earth's surface (Fig. 10.1). In these equations $R_{l,t} = c_{l,t}t$. Near epicentre ($r_{l,t} \rightarrow 0$) the width of the seismic spot $\Delta r_{l,t} \simeq \sqrt{2z_0 l_{l,t}}$ is much larger than $l_{l,t}$ ($l_{l,t} \ll z_0$). The distance $\sqrt{2z_0 l_{l,t}}$ defines the epicentral region. From equations (10.2) we get the velocities $v_{l,t} = dr_{l,t}/dt = c_{l,t}\frac{R_{l,t}}{r_{l,t}}$ of the seismic spot on Earth's surface. We can see that these velocities are greater than the velocities of the elastic waves. For regions of interest we approximate the Earth by a homogeneous and isotropic half-space with a plane free surface.

In the above formulae the velocities $c_{l,t}$ are global average velocities. The average is made over the propagation region with a radius R. It follows that the average velocities differ for different radii R. Therefore, they may exhibit small, local variations on Earth's surface, at points placed at different distances from the focus.

The displacement produced by the main shock on Earth's surface is given by

$$u_r = \frac{\chi_0 r}{4c_l}\frac{\tau}{(c_l^2\tau^2-r^2)^{3/2}} \ ,$$

$$u_\varphi = -\frac{h_{0z} r}{4c_t}\frac{\tau}{(c_t^2\tau^2-r^2)^{3/2}} \ , \qquad (10.4)$$

$$u_z = \frac{h_{0\varphi}}{4c_t r}\frac{c_t^2\tau^3}{(c_t^2\tau^2-r^2)^{3/2}} \ ,$$

for $c_{l,t}\tau > r$, where r is the distance from epicentre to the observation point on Earth's surface, $\tau = t(1-\varepsilon)$, $\varepsilon = R/r-1$ and the potentials χ_0 and h are of the order $M/\rho R$, where M is the magnitude of the seismic moment (elastic energy released in the focus). The coordinates r, φ, z are cylindrical coordinates. The time τ in equations (10.4) is measured from the epicentre. We note that the main shock (also called secondary waves) moves with velocities $c_{l,t}$, which are smaller than the velocities $v_{l,t} = c_{l,t}R/r$ of the P and S waves on Earth's surface. The main shock moves behind the P and S waves. Equations (10.4) are valid for $\varepsilon < 1$ and within a limited range of the order z_0 for distances r, centered on a distance of the order z_0, where z_0 is the depth of the focus ($z_0/\sqrt{3} < r < 2z_0$). The singularity at $c_{l,t}\tau = r$ is smoothed out according to the replacement $c_{l,t}^2\tau^2 - r^2 \mid_{c_{l,t}\tau=r}\rightarrow r^2\varepsilon$.[2] The main shock has the appearance of a seismic "wall" with a long tail. We may use the simplified formulae

$$u_r = \frac{Mr}{4\rho c_l R}\frac{\tau}{(c_l^2\tau^2-r^2)^{3/2}} \ ,$$

$$u_\varphi = -\frac{Mr}{4\rho c_t R}\frac{\tau}{(c_t^2\tau^2-r^2)^{3/2}} \ , \qquad (10.5)$$

$$u_z = \frac{M}{4\rho c_t r R}\frac{c_t^2\tau^3}{(c_t^2\tau^2-r^2)^{3/2}} \ ,$$

for the main-shock displacement, where the term ε is neglected in τ.

10.2 Seismic spectrum

The soil layers and the buildings on Earth's surface exhibit vibration eigenfrequencies. Therefore, we are interested in the frequency contents of the local seismic displacement (local velocity, acceleration),

[2]B. F. Apostol, loc. cit.

in order to avoid resonances. The frequency content is given by the Fourier transform. The Fourier transform of the seismic displacement, velocity and acceleration is also called the site spectral response. Let us perform the time Fourier transform of the function $\delta'(t-R/c_l)$. Formally, it is given by

$$f(\omega)=\int dt\delta'(t-R/c_l)e^{i\omega t}\ . \tag{10.6}$$

According to the hypotheses used in deriving equations (10.1), the function $\delta(t-R/c_l)$ for fixed R should be viewed as $1/\Delta t$ in the range $-\Delta t/2$ to $\Delta t/2$, centered on $t=R/c_l$. The temporal range is $\Delta t=l/c_l$, where l is of the order of the dimension of the seismic focus (localization of the tensorial force which generates the seismic waves). We choose the origin of time at $t=R/c_l$. Consequently, equation (10.6) leads to

$$f_P(\omega)=\int_{-\Delta t/2}^{\Delta t/2}dt\delta'(t)\sin\omega t=-\frac{2c_l}{l}\sin\omega l/2c_l \tag{10.7}$$

and

$$\boldsymbol{u}_P(\omega)\simeq\frac{TM_4}{2\pi\rho c_l^2 lR}\boldsymbol{n}\sin\omega l/2c_l\ . \tag{10.8}$$

Since the amplitude of this function is not appropriate for giving relevant information in empirical studies we may replace l in the amplitude by c_lT, such that[3]

$$\boldsymbol{u}_P(\omega)\simeq\frac{M_4}{2\pi\rho c_l^3 R}\boldsymbol{n}\sin\omega l/2c_l\ . \tag{10.9}$$

The corresponding Fourier transforms of the velocity and acceleration are $\boldsymbol{v}_P(\omega)=\omega\boldsymbol{u}_P(\omega)$ and $\boldsymbol{a}_P(\omega)=-\omega^2\boldsymbol{u}_P(\omega)$. The function $\sin\omega l/2c_l$ (for displacement) has a maximum value for

$$\omega_m\simeq\pi c_l/l \tag{10.10}$$

(period $T_m\simeq 2l/c_l$). The velocity spectrum has a maximum at $\omega_m\simeq 4c_l/l$ and the acceleration spectrum has a maximum at $\omega_m\simeq 3\pi c_l/2l$

[3] B. F. Apostol, "Seismic spectrum and spectral response", J. Theor. Phys. **335** (2022).

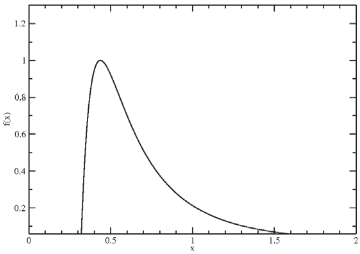

Figure 10.2: Typical acceleration spectrum $f(x) = \frac{1}{x^2}\sin\frac{1}{x}$ as a function of period, exhibiting the maximum at $x \simeq 3\pi/4$.

(period $\simeq 4l/3c_l$). By using these maxima values we can obtain an estimation of the dimension l of the focus (for known velocities $c_{l,t}$). The maximum value of the functions gives an estimation of the parameter TM_4/ρ.

The parameters of the earthquakes and the seismic sources can be determined from the displacement produced by the seismic waves on Earth's surface.[4] For instance, for $l = 100m$ (earthquake magnitude $\simeq 5$) and a mean velocity $c = 5km/s$ ($c_l = 7km/s$, $c_t = 3km/s$) we get $T_m = 0.04s$ (from the maximum of the displacement). An approximate estimation of the parameter l is given by $E \simeq 2\rho c^2 l^3$, where E is the earthquake energy; making use of the Hanks-Kanamori relationship $\lg E = \frac{3}{2}M_w + 15.6$, where M_w is the moment magnitude (and E in erg), we get the approximate formula $\lg l \simeq \frac{1}{2}M_w + 1$ ($\rho = 5.5g.cm^3$, $c = 5km/s$).[5] This estimation differs from the exact determination of l and T, because of the average procedure. A graphical representation of the function $(1/x^2)\sin(1/x)$, which is a typical acceleration spectrum as a function of the period, is given in Fig. 10.2

By a similar procedure we compute the Fourier transform of the S wave. The S wave has a time delay $\delta = R/c_t - R/c_l$ with respect to

[4] B. F. Apostol, "An inverse problem in Seismology: derivation of the seismic source parameters from P and S seismic waves", J. Seismol. **23** 1017 (2019).
[5] B. F. Apostol, *Seismology*, Nova, NY (2020).

the P wave. Consequently, the Fourier transform is given by

$$f_S(\omega) = \cos\omega\delta \int_{-\Delta t/2}^{\Delta t/2} dt\delta'(t)\sin\omega t = -\frac{2c_t}{l}\cos\omega\delta\sin\omega l/2c_t \quad (10.11)$$

(where $\Delta t = l/c_t$). This function has a maximum at $\omega_m \simeq \pi c_t/l$, which implies a shift

$$\Delta T_m \simeq 2l\frac{c_l - c_t}{c_l c_t} \quad (10.12)$$

in its period, with respect to the P wave.

We pass now to the Fourier transforms of the main shock (equations (10.4)). The time delay R/c_l, with respect to the P wave, can be left aside. We have two types of Fourier transforms, written for a generic velocity c:

$$\begin{aligned} g(\omega) &= Re\int_{r/c}^{\infty} d\tau \frac{\tau}{(c^2\tau^2-r^2)^{3/2}}e^{i\omega\tau} \ , \\ h(\omega) &= Re\int_{r/c}^{\infty} d\tau \frac{\tau^3}{(c^2\tau^2-r^2)^{3/2}}e^{i\omega\tau} \ . \end{aligned} \quad (10.13)$$

The first Fourier transform is

$$g(\omega) = \frac{1}{c^2 r\sqrt{\varepsilon}}\cos(\omega r/c) - \frac{\pi\omega}{2c^3}ReH_0^{(1)}(\omega r/c) \ , \quad (10.14)$$

where $H_0^{(1)}$ is the Hankel function of the first kind and zeroth order and $\varepsilon = c\tau/r - 1$.[6] For all frequencies and distances of interest $H_0^{(1)}(\omega r/c) \simeq \sqrt{\frac{2c}{\pi\omega r}}e^{i(\omega r/c - \pi/4)}$; this function and the function $\cos(\omega r/c)$ are rapidly varying functions; they are filtered out in the seismic spectrum. Similarly, the Fourier transform $h(\omega)$ implies rapidly varying trigonometric functions and the Hankel function of the first kind and the first order $H_1^{(1)}$. It follows that the seismic spectrum reduces to the contribution of the P and S seismic waves.

10.3 Local frame

We use a local frame with the origin O at the observation point and axes 1, 2, 3 corresponding to the directions North-South (NS), West-

[6] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products*, 6th ed., Academic Press, NY (2000), p. 904, 8.421 (1).

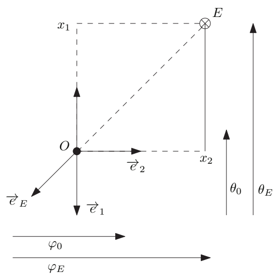

Figure 10.3: Local frame at the observation point O. The unit vectors are $\boldsymbol{e}_1$ for NS, $\boldsymbol{e}_2$ for WE and $\boldsymbol{e}_3$ for the vertical coordinate z. $x_{1,2}$ are the epicentre (E) coordinates.

East (WE) and the local vertical (z). Let θ_0 and φ_0 be the latitude and the longitude of the origin O and θ_E and φ_E the latitude and the longitude of the epicentre E. The coordinates of the epicentre are

$$x_1 = -R_0 \Delta\theta \ , \ x_2 = R_0 \cos\theta_E \Delta\varphi \ , \tag{10.15}$$

where $\Delta\theta = \theta_E - \theta_0$, $\Delta\varphi = \varphi_E - \varphi_0$ and R_0 is the Earth's radius (Fig. 10.3). We introduce the unit vector from the epicentre to the observation point

$$\boldsymbol{e}_E = -\frac{x_1}{\sqrt{x_1^2 + x_2^2}}\boldsymbol{e}_1 - \frac{x_2}{\sqrt{x_1^2 + x_2^2}}\boldsymbol{e}_2 \ , \tag{10.16}$$

such that the unit vector from the focus to the observation point is

$$\boldsymbol{n} = \frac{r}{R}\boldsymbol{e}_E + \frac{z_0}{R}\boldsymbol{e}_3 \ , \tag{10.17}$$

where $\boldsymbol{e}_{1,2,3}$ are the unit vectors along the local axes (NS, WE, z; Fig. 10.4). Also, we introduce the notations

$$\begin{aligned} \alpha &= -\frac{x_1}{\sqrt{x_1^2+x_2^2}} = \frac{\Delta\theta}{\sqrt{\Delta\theta^2+\cos^2\theta_E\Delta\varphi^2}} \ , \\ \beta &= -\frac{x_2}{\sqrt{x_1^2+x_2^2}} = -\frac{\cos\theta_E\Delta\varphi}{\sqrt{\Delta\theta^2+\cos^2\theta_E\Delta\varphi^2}} \ . \end{aligned} \tag{10.18}$$

The displacement of the P wave (equations (10.1)) can be written as

$$\begin{aligned}\boldsymbol{u}_P &= -\frac{TM_4}{4\pi\rho c_l^3 R}\boldsymbol{n}\delta'(t-R/c_l) = \\ &= -\frac{TM_4}{4\pi\rho c_l^3 R}\left(\frac{\alpha r}{R}\boldsymbol{e}_1 + \frac{\beta r}{R}\boldsymbol{e}_2 + \frac{z_0}{R}\boldsymbol{e}_3\right)\delta'(t-R/c_l)\ .\end{aligned} \tag{10.19}$$

We write $\boldsymbol{M} = M_4(a\boldsymbol{e}_1 + b\boldsymbol{e}_2 + c\boldsymbol{e}_3)$ and get $c = \frac{R}{z_0}\left(1 - a\frac{\alpha r}{R} - b\frac{\beta r}{R}\right)$ from the condition $\boldsymbol{Mn} = M_4$. Consequently, the displacement of the S wave (equations (10.1)) can be written as

$$\begin{aligned}\boldsymbol{u}_S &= -\frac{T(M_4\boldsymbol{n}-\boldsymbol{M})}{4\pi\rho c_t^3 R}\delta'(t-R/c_t) = \\ &= -\frac{TM_4}{4\pi\rho c_t^3 R}\{\left(\frac{\alpha r}{R} - a\right)\boldsymbol{e}_1 + \left(\frac{\beta r}{R} - b\right)\boldsymbol{e}_2 + \\ &+ \left[\frac{z_0}{R} - \frac{R}{z_0}\left(1 - \frac{\alpha r}{R}a - \frac{\beta r}{R}b\right)\right]\boldsymbol{e}_3\}\delta'(t-R/c_t)\ .\end{aligned} \tag{10.20}$$

It follows that the displacements along the local directions are

$$\begin{aligned}u_1(NS) &= -\frac{TM_4}{4\pi\rho R}[\frac{\alpha r}{c_l^3 R}\delta'(t-R/c_l) + \\ &+ \frac{1}{c_t^3}\left(\frac{\alpha r}{R} - a\right)\delta'(t-R/c_t)]\ , \\ u_2(WE) &= -\frac{TM_4}{4\pi\rho R}\frac{\beta r}{c_l^3 R}\delta'(t-R/c_l) + \\ &+ \frac{1}{c_t^3}\left(\frac{\beta r}{R} - b\right)\delta'(t-R/c_t)]\ , \\ u_3(z) &= -\frac{TM_4}{4\pi\rho R}\{\frac{z_0}{c_l^3 R}\delta'(t-R/c_l) + \\ &+ \frac{1}{c_t^3}\left[\frac{z_0}{R} - \frac{R}{z_0}\left(1 - \frac{\alpha r}{R}a - \frac{\beta r}{R}b\right)\right]\delta'(t-R/c_t)\}\ .\end{aligned} \tag{10.21}$$

We can see that the spectrum of each of these components exhibits, in general, two maxima. The position of these maxima depends on the coefficients of the functions $\delta'(t-R/c_{l,t})$, *i.e.* the position of the observation point (α, β, r, R) and the orientation of the seismic moment vector (coefficients a and b). The fitting of the curves given

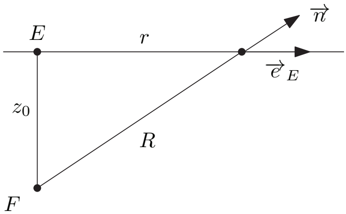

Figure 10.4: The unit vectors $\boldsymbol{n}$ and $\boldsymbol{e}_E$ from the focus and from the epicentre to the observation point.

by equation (10.21) to the seismic records gives access, in principle, to the remaining parameters M_4, a and b.

A great simplification of the equations (10.21) is obtained for the particular values $\alpha = \beta = 1/\sqrt{2}$ and $r = z_0 = R/\sqrt{2}$ (they correspond approximately to data recorded in Bucharest, or Cernavoda, for Vrancea intermediate-depth earthquakes). Also, we can replace the velocities $c_{l,t}$ by a local effective velocity c. The displacement spectrum becomes

$$\begin{aligned} u_1(\omega) &= \tfrac{M_4}{2\pi\rho c^3 R}(1-a)\sin\tfrac{\omega l}{2c} \ , \\ u_2(\omega) &= \tfrac{M_4}{2\pi\rho c^3 R}(1-b)\sin\tfrac{\omega l}{2c} \ , \\ u_3(\omega) &= \tfrac{M_4}{2\sqrt{2}\pi\rho c^3 R}(a+b)\sin\tfrac{\omega l}{2c} \ . \end{aligned} \tag{10.22}$$

The position of the maximum ω_m, of the order c/l, depends on the earthquake magnitude, through l, and the local conditions, through the parameter c.

We can compare the seismic spectrum (for example, the Fourier transform of the acceleration) for two earthquakes, 1 and 2, recorded at the same site. Then, the effective velocity c is the same. From the two distinct positions of the maxima, $\omega_{m1} \simeq \frac{3\pi c}{2l_1}$ and $\omega_{m2} \simeq \frac{3\pi c}{2l_2}$ (for accelerations), we can derive the ratio of the dimensions of the foci $l_1/l_2 = \omega_{m2}/\omega_{m1}$. The ratio of the maximum amplitudes gives ratios of the type $M_{41}(1-a_1)/M_{42}(1-a_2)$ (for the NS component). Also, we can compare the spectrum for the same earthquake and two distinct sites. In this case the parameter l is the same and we get the ratio of the two effective velocities $c_1/c_2 = \omega_{m1}/\omega_{m2}$. If we know one velocity (for example the standard mean velocity $c = 5km/s$), we can get the other velocity. Also, we can characterize the local conditions (velocities c) with respect to a standard site (*e.g.*, a bedrock). The ratio of the maximum amplitudes gives the ratio $\rho_1 c_1^3 R_1/\rho_2 c_2^3 R_2$ (for any component).

10.4 Spectral response

The constructions built on Earth's surface and, in general, finite-size irregularities, may be approximated, in a simplified picture, by linear harmonic oscillators. The motion of such an oscillator is governed by the equation

$$m\ddot{u} + m\omega_0^2 u + m\gamma\dot{u} = ma(t) \ , \tag{10.23}$$

where m is the mass of the oscillator, u is its displacement, ω_0 is its eigenfrequency, γ is a damping coefficient and $a(t)$ is the acceleration generated by the action of an external force (a coupling impedance can also be introduced). We adopt an external acceleration $a(t) = a_0(\omega)\sin\omega t$, given by the seismic spectrum calculated above (site response). The solution of this equation is

$$\begin{gathered} u(t) = -a_0(\omega)\tfrac{\omega^2-\omega_0^2}{(\omega^2-\omega_0^2)^2+\omega^2\gamma^2}\sin\omega t- \\ -a_0(\omega)\tfrac{\omega\gamma}{(\omega^2-\omega_0^2)^2+\omega^2\gamma^2}\cos\omega t \ . \end{gathered} \tag{10.24}$$

We can see that it exhibits a resonance for $\omega = \omega_0$, attenuated by the damping. A large damping occurs for nonlinear structures. At resonance, $u(t) = -\frac{a_0(\omega_0)}{\omega_0\gamma}\cos\omega_0 t$. The ratio $\mid u(\omega_0)/a_0(\omega_0) \mid$, where $u(\omega_0)$ is the Fourier transform of $u(t)$, leads to an amplification factor $1/\omega_0\gamma$.

The energy conservation resulted from equation (10.23) is

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{u}^2 + \frac{1}{2}m\omega^2 u^2\right) + m\gamma\dot{u}^2 = F\dot{u} \ , \tag{10.25}$$

where $\mathcal{E} = \frac{1}{2}m\dot{u}^2 + \frac{1}{2}m\omega^2 u^2$ is the energy of the oscillator, $W = m\gamma\dot{u}^2$ is the energy dissipated per unit time and $F\dot{u}$ is the work done by the force $F(t) = ma(t)$ per unit time. The oscillator receives energy from the external source and dissipates it, partially. The average energy of the oscillator is

$$\overline{\mathcal{E}} = \frac{1}{2}ma_0^2\frac{\omega^2+\omega_0^2}{(\omega^2-\omega_0^2)^2+\omega^2\gamma^2} \tag{10.26}$$

and the average energy dissipated per unit time is

$$\overline{W} = \overline{m\gamma\dot{u}^2} = \frac{1}{2}ma_0^2\frac{\gamma\omega^2}{(\omega^2-\omega_0^2)^2+\omega^2\gamma^2}\ . \tag{10.27}$$

At resonance $\overline{\mathcal{E}} = ma_0^2/\gamma^2$ and $\overline{W} = ma_0^2/2\gamma$. For small γ ($\gamma \ll \omega_0$) the energy dissipated in a period is $ma_0^2/2\gamma\omega_0 \ll \overline{E}$; it is much smaller than the oscillator energy. On the contrary, for a large damping the dissipated energy is $ma_0^2/2\gamma^2$; we can see that it is comparable to the energy of the oscillator. It follows that a small dissipation ($\gamma \ll \omega_0$) is preferrable, if we cannot avoid resonance, although the displacement, velocity and acceleration (amplification factor) are higher.

For a full description of the seismic response the acceleration $a_0(\omega)$ resulted from the seismic spectrum should be introduced in equation (10.24). According to equation (10.8) we can use

$$a_0(\omega) = b\omega^2\sin\omega l/2c \tag{10.28}$$

with a generic velocity c, where b is of the order $M/\rho c^3R$. Far from resonance the response of the oscillator does not differ appreciably from the seismic spectrum (site response); the maximum frequency remains at $\omega_m \simeq \pi c/l$ (for displacement). On the contrary, close to resonance the response is given approximately by

$$u(t) \simeq -\frac{b}{4}\frac{\omega_0\gamma}{(\omega-\omega_0)^2+\gamma^2/4}\sin\omega_0 l/2c\cos\omega_0 t\ , \tag{10.29}$$

whence we can see that the response is maximal for the resonance frequency ω_0.

An elastic, vertical, thin bar, with length L and with the lower end embedded in the ground on Earth's surface and the upper end free, has the eigenfrequencies $\omega_n = \frac{(2n+1)\pi}{2}\frac{c}{L}$, where c is the velocity of the elastic waves in the bar (either longitudinal, or transverse) and

$n = 0, 1, 2....$[7] The seismic acceleration has a narrow peak for $\omega_m \simeq \frac{3\pi}{2}\frac{c_{l,t}}{l}$. It follows that the resonance is avoided for $L \neq (2n+1)\frac{c}{3c_{l,t}}l$.

Task #10 of Practical Seismology consists in estimating the ratio l/c from the position ω_m (frequency) of the maxima exhibited by the site spectral response ($\omega_m = \frac{\pi c}{l}, \frac{4c}{l}, \frac{3\pi c}{2l}$ for displacement, velocity and acceleration); this way, for a known velocity c, we have an estimate of the dimension of the seismic focus l. A comparison between two earthquakes for the same site leads to the ratios $l_1/l_2 = \omega_{m2}/\omega_{m1}$, while a comparison between two sites for the same earthquake gives the ratios $c_1/c_2 = \omega_{m1}/\omega_{m2}$.

[7] B. F. Apostol, *The Theory of Earthquakes*, Cambridge International Science Publishing, Cambridge (2017).

11 Oscillator and Elastic Waves

11.1 Amplification factors

Finite size inhomogeneities in elastic media can be viewed as localized oscillators, under the action of the elastic waves. We view the Earth as a homogeneous and isotropic elastic half-space which sustains the propagation of the seismic waves. The wavelengths of the seismic waves are longer than the dimension of the seismic focus and the dimension of many inhomogeneities embedded in Earth or placed on Earth's surface. These inhomogeneities are themselves elastic media capable of sustaining their own elastic vibrations. Therefore, they exhibit vibration eigenfrequencies, such that they may be viewed as oscillators.

11.1.1 Damped harmonic oscillator

Let a particle of mass m and coordinate $x(t)$ be subjected to an elastic force $-kx$, a friction force $-\alpha\dot{x}$ and an external force $f(t)$; k denotes the elastic force constant and α is the friction coefficient. The corresponding equation of motion reads

$$m\ddot{x} + kx + \alpha\dot{x} = f \ . \tag{11.1}$$

We introduce the eigenfrequency ω_0 given by $\omega_0^2 = k/m$ and look for a solution $x = \xi e^{-\lambda\omega_0 t}$. For $\alpha/2m\omega_0 = \lambda$ we obtain

$$\ddot{\xi} + \omega_0^{'2}\xi = (f/m)e^{\lambda\omega_0 t} \ , \tag{11.2}$$

where $\omega_0^{'} = \omega_0(1-\lambda^2)^{1/2}$. We introduce $\eta = \dot{\xi} + i\omega_0^{'}\xi$ which obeys the equation

$$\dot{\eta} - i\omega_0^{'}\eta = (f/m)e^{\lambda\omega_0 t} \ , \tag{11.3}$$

and look for a solution $\eta = ue^{i\omega_0^{'} t}$. We find the new equation

$$\dot{u} = (f/m)e^{-i\omega_0^{'} t + \lambda\omega_0 t} \ , \tag{11.4}$$

whose solution is

$$u = \int_0^t d\tau (f/m) e^{-i\omega_0^{'}\tau + \lambda\omega_0\tau} + u_0 \ , \tag{11.5}$$

where u_0 is the initial condition. It follows

$$\eta = \int_0^t d\tau (f/m) e^{i\omega_0^{'}(t-\tau) + \lambda\omega_0\tau} + u_0 e^{i\omega_0^{'} t} \ , \tag{11.6}$$

and $\xi = (1/\omega_0^{'}) Im\eta$, *i.e.*

$$\xi = \frac{1}{\omega_0^{'}} \int_0^t d\tau (f/m) e^{\lambda\omega_0\tau} \sin\omega_0^{'}(t-\tau) + \frac{1}{\omega_0^{'}} |u_0| \sin(\omega_0^{'} t + \varphi) \ , \tag{11.7}$$

where φ is an initial phase. Finally we obtain the coordinate

$$\begin{gathered} x = \tfrac{1}{\omega_0^{'}} \int_0^t d\tau (f/m) e^{-\lambda\omega_0(t-\tau)} \sin\omega'_0(t-\tau) + \\ + \tfrac{1}{\omega_0^{'}} |u_0| \, e^{-\lambda\omega_0 t} \sin(\omega_0^{'} t + \varphi) \ . \end{gathered} \tag{11.8}$$

We choose $u_0 = 0$ and get the forced oscillations with attenuation

$$x = \frac{1}{\omega_0^{'}} \int_0^t d\tau (f/m) e^{-\lambda\omega_0(t-\tau)} \sin\omega_0^{'}(t-\tau) \tag{11.9}$$

and

$$\begin{gathered} \dot{x} = \tfrac{1}{\omega_0^{'}} \int_0^t d\tau (f/m) e^{-\lambda\omega_0(t-\tau)} \cdot \\ \cdot [-\lambda\omega_0 \sin\omega_0^{'}(t-\tau) + \omega_0^{'} \cos\omega_0^{'}(t-\tau)] \ , \end{gathered} \tag{11.10}$$

which both satisfy the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$. We get also

$$\begin{gathered} \ddot{x} = f/m - \\ -\tfrac{1}{\omega_0^{'}} \int_0^t d\tau (f/m) e^{-\lambda\omega_0(t-\tau)} [(1-2\lambda^2)\omega_0^2 \sin\omega_0^{'}(t-\tau) + \\ + 2\lambda\omega_0\omega_0^{'} \cos\omega_0^{'}(t-\tau)] \ , \end{gathered} \tag{11.11}$$

for acceleration, which equals the external acceleration $\ddot{x}(0) = f(0)/m$ at the initial moment of time. Usually, the damping parameter λ is small ($\lambda \ll 1$), so that we may replace $\omega_0^{'}$ by ω.

11.1.2 Periodic external force

With the notations introduced above we may rewrite equation (11.1) as

$$\ddot{x} + \omega_0^2 x + 2\lambda\omega_0\dot{x} = f/m \ , \tag{11.12}$$

and assume a periodic external force as given by

$$f = f_0 \cos\omega t \ . \tag{11.13}$$

The solution $x(t)$ of the equation (11.12) is obtained from equation (11.9) by introducing this force, as given by equation (11.13) (for vanishing initial conditions). The same solution is also obtained as $x = x_0 + x_1$, where x_0 is the solution of the homogeneous equation and x_1 is a particular solution of the inhomogeneous equation. It is easy to check that x_0 is given by $x_0 = ae^{-\lambda\omega_0 t}\cos(\omega_0^{'}t + \alpha)$, where the amplitude a and the phase α are not yet determined. The particular solution of the inhomogeneous equation reads $x_1 = b\cos(\omega t + \beta)$, where

$$\begin{aligned} b &= \frac{f_0/m}{\sqrt{(\omega^2-\omega_0^2)^2+4\lambda^2\omega^2\omega_0^2}} \ , \\ \tan\beta &= \frac{2\lambda\omega\omega_0}{\omega^2-\omega_0^2} \ . \end{aligned} \tag{11.14}$$

The phase β is always negative, $-\pi < \beta < 0$, *i.e.*

$$\begin{aligned} \sin\beta &= -2\lambda\omega\omega_0/\sqrt{(\omega^2-\omega_0^2)^2+4\lambda^2\omega^2\omega_0^2} \ , \\ \cos\beta &= (\omega_0^2-\omega^2)/\sqrt{(\omega^2-\omega_0^2)^2+4\lambda^2\omega^2\omega_0^2} \ , \end{aligned} \tag{11.15}$$

so that the particle lags always behind the external force. The amplitude b is maximal for $\omega = \omega_0(1-2\lambda^2)^{1/2}$. For $\lambda \ll 1$ the resonance occurs for $\omega = \omega_0$. Let $\omega = \omega_0 + \varepsilon$; then

$$\begin{aligned} b &= \frac{f_0/2m\omega_0}{\sqrt{\varepsilon^2+\lambda^2\omega_0^2}} \ , \\ \tan\beta &= \lambda\omega_0/\varepsilon \ . \end{aligned} \tag{11.16}$$

In the absence of the damping the phase of the oscillation undergoes a jump at resonance (b changes the sign), while the damping smooths this jump out ($\beta = -\pi/2$ at resonance).

The friction force $-\alpha\dot{x}$ can be derived from $-\partial F/\partial\dot{x}$ where $F = (1/2)\alpha\dot{x}^2$ is the dissipation function. It follows that the Euler-Lagrange equations reads

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{x}} = \frac{\partial L}{\partial x} - \partial F/\partial\dot{x} \ , \qquad (11.17)$$

where L is the Lagrange's function. The energy E changes in time according to

$$\begin{aligned} dE/dt = \tfrac{d}{dt}(\dot{x}\partial L/\partial\dot{x} - L) = \\ = \dot{x}[\tfrac{d}{dt}(\partial L/\partial\dot{x}) - \partial L/\partial x] = -\dot{x}\partial F/\partial\dot{x} = -2F \ . \end{aligned} \qquad (11.18)$$

For a time long enough the motion is stabilized, *i.e.* $x \simeq x_1$ and the energy is constant. The particle absorbs continuously energy from the external force and dissipates it through friction. The dissipated average energy per unit time is given by

$$I(\omega) = 2\overline{F} = 2m\lambda\omega_0\omega^2 b^2\overline{\sin^2(\omega t + \beta)} = m\lambda\omega_0\omega^2 b^2 \ , \qquad (11.19)$$

and close to the resonance

$$I(\omega_0) = \frac{f_0^2}{4m}\frac{\lambda\omega_0}{\varepsilon^2 + \lambda^2\omega_0^2} \ , \qquad (11.20)$$

which is a dispersive function of the frequency ε. Its integral does not depend on frequency, $\int_0^\infty d\omega_0 I(\omega_0) = \pi f_0^2/4m$.

We now turn back to the general solution

$$x = ae^{-\lambda\omega_0 t}\cos(\omega_0 t + \alpha) + b\cos(\omega t + \beta) \ , \qquad (11.21)$$

where we neglect the small effect of the damping on the frequency, *i.e.* $\omega_0^{'} \simeq \omega_0$, and impose the vanishing initial conditions $x(0) = \dot{x}(0) = 0$. Making use of equations (11.14) and (11.15) we obtain

$$a = \frac{f_0}{m}\frac{\sqrt{(\omega^2 - \omega_0^2)^2 + \lambda^2(\omega^2 + \omega_0^2)^2}}{(\omega^2 - \omega_0^2)^2 + 4\lambda^2\omega^2\omega_0^2} \qquad (11.22)$$

and

$$\tan\alpha = -\frac{\lambda(\omega^2+\omega_0^2)}{\omega_0^2-\omega^2} \ . \tag{11.23}$$

At resonance the phase α passes through $\pi/2$, *i.e.*

$$\begin{aligned} \sin\alpha &= \lambda(\omega^2+\omega_0^2)/\sqrt{(\omega^2-\omega_0^2)^2+\lambda^2(\omega^2+\omega_0^2)^2} \ , \\ \cos\alpha &= -(\omega_0^2-\omega^2)/\sqrt{(\omega^2-\omega_0^2)^2+\lambda^2(\omega^2+\omega_0^2)^2} \ . \end{aligned} \tag{11.24}$$

Also at resonance the amplitude

$$a = f_0/2m\lambda\omega_0^2 \tag{11.25}$$

equals the amplitude $b = f_0/2m\lambda\omega_0^2$ as given by equation (11.16).

We also establish now the coordinates for a motion of the particle proceeding solely under the action of the external force f. Since $f = f_0\cos\omega t$ it follows that acceleration is $ac = (f_0/m)\cos\omega t$, which has a maximum value $ac_{max} = f_0/m$. The velocity is given by $v = (f_0/m\omega)\sin\omega t$, which initially vanishes and has a maximum value $v_{max} = f_0/m\omega$. Finally, the displacement d is given by

$$d = -(f_0/m\omega^2)\cos\omega t + (f_0/m\omega^2) = (2f_0/m\omega^2)\sin^2(\omega t/2) \ , \tag{11.26}$$

for a vanishing initial displacement; its maximum value is $d_{max} = 2f_0/m\omega^2$.

11.1.3 Amplification factors at resonance

According to the results derived above at resonance $a = b = f_0/2m\lambda\omega_0^2$, $\alpha = \pi/2$ and $\beta = -\pi/2$. Then, the general solution given by equation (11.21) becomes

$$x = \frac{f_0}{2m\lambda\omega_0^2}(1-e^{-\lambda\omega_0 t})\sin\omega_0 t \ . \tag{11.27}$$

We look for the local minima of this function, *i.e.* the solutions of the equation

$$\tan\omega_0 t = -\frac{1}{\lambda}(e^{\lambda\omega_0 t}-1) \ ; \tag{11.28}$$

they are close to

$$\omega_0 t = (2k+1)\pi/2 \tag{11.29}$$

(and slightly above), where $k = 0, 1, 2....$ The maximum values of the coordinate modulus are given by

$$|x|_{max} \simeq \frac{f_0}{2m\lambda\omega_0^2}(1 - e^{-\lambda(2k+1)\pi/2}) \ . \tag{11.30}$$

The amplification factor of the displacement is defined as the ratio

$$F_d = |x|_{max}/d_{max} \simeq \frac{1}{4\lambda}(1 - e^{-\lambda(2k+1)\pi/2}) \ . \tag{11.31}$$

For small values of the damping coefficient λ the amplification factor may attain considerably higher-than-unity values. Indeed, for $\lambda(2k+1)\pi/2 \ll 1$ we get

$$F_d \simeq (2k+1)\pi/8 \tag{11.32}$$

from equation (11.31). Typical values for λ allow the integer k to go up to $k = 1, 2, 3, 4$, where the amplification factor reaches the values 1.18, 1.96, 2.75 and 3.53, respectively, for times $t = (2k+1)T_0/4$, where T_0 is the eigenperiod of the oscillations. For higher values of the damping ($\lambda > 0.25$, for instance) the amplification factor is less than unity.[1]

A similar analysis holds for the velocity

$$\dot{x} = \frac{f_0}{2m\lambda\omega_0}[\lambda e^{-\lambda\omega_0 t}\sin\omega_0 t + (1 - e^{-\lambda\omega_0 t})\cos\omega_0 t] \ , \tag{11.33}$$

which reaches the maximum modulus values

$$|\dot{x}|_{max} \simeq \frac{f_0}{2m\lambda\omega_0}(1 - e^{-\lambda k\pi}) \tag{11.34}$$

for $\omega_0 t \simeq k\pi$, $k = 1, 2, 3...$(the maximum placed between 0 and $\pi/2$ is left aside). The amplification factor for velocities is defined by

$$F_v = |\dot{x}|_{max}/v_{max} \simeq \frac{1}{2\lambda}(1 - e^{-\lambda k\pi}) \ . \tag{11.35}$$

[1] B. F. Apostol, "Amplification factors in oscillatory motion", Roum. J. Phys. **49** 691 (2004).

For small values of the damping coefficient the amplification factor is given by

$$F_v \simeq k\pi/2 \; ; \tag{11.36}$$

it may attain higher values than the amplification factor for displacement (up to 2π for instance, corresponding to $k = 4$).

Within the approximation $\lambda\omega_0 t \ll 1$ and $\lambda \ll 1$ employed here the acceleration can be written as

$$\ddot{x} = \frac{f_0}{2m}(\omega_0 t \sin\omega_0 t - 2\cos\omega_0 t) \; ; \tag{11.37}$$

its modulus attains maximum values for $\omega_0 t$ satisfying the equation $\tan\omega_0 t = -\omega_0 t/3$. The approximate solutions of this equations, corresponding to higher values of the acceleration, are given by $\omega_0 t \simeq (2k+1)\pi/2$ for $k = 2, 3....$ The amplification factor for acceleration is defined by

$$F_a = |\ddot{x}|_{max}/ac_{max} = \frac{1}{2}|\omega_0 t \sin\omega_0 t - 2\cos\omega_0 t|_{max} \; . \tag{11.38}$$

Its maximum values are (slightly less than) $F_a \simeq (2k+1)\pi/4$.

Far from resonance the amplification factors decrease. It is worth noting that the amplification factors are higher than unity because of the large amplitudes of oscillations at resonance, and far from resonance these amplitudes decrease according to equations (11.14) and (11.22), and, consequently, the amplification factors decrease too. Indeed, it is worth analyzing the energy forced into the oscillating particle for zero damping. From equation (11.6) we obtain

$$\eta = e^{i\omega_0 t}\int_{-\infty}^{+\infty} d\tau (f/m) e^{-i\omega_0 \tau} \; , \tag{11.39}$$

in this case, for a very large duration t and vanishing initial conditions at $t \to -\infty$. On the other hand the energy of the particle is $E = m(\dot{x}^2 + \omega_0^2 x^2)/2 = m|\eta|^2/2$, *i.e.*

$$E = \frac{1}{2m}\left|\int_{-\infty}^{+\infty} d\tau f e^{-i\omega_0\tau}\right|^2 \; , \tag{11.40}$$

which shows that the pumped energy is associated with the Fourier component of the external force corresponding to the particle frequency. For $f = f_0 \cos\omega t$ we obtain $E = \pi^2 f_0^2 \delta^2(\Delta\omega)/2m$ close to resonance, where $\Delta\omega = \omega - \omega_0$. The energy per unit time leads to $I = E/t = \pi f_0^2 \delta(\Delta\omega)/4m$, since $t\Delta\omega = 2\pi$; it may also be written as

$$I = \frac{f_0^2}{4m} \frac{\lambda\omega_0}{(\Delta\omega)^2 + \lambda^2\omega_0^2} \ , \tag{11.41}$$

where λ is a vanishing parameter, which coincides with the dissipated energy per unit time given by (11.20). The latter equation shows that the energy absorbed from the external force equals the dissipated energy and it has a maximum value at resonance.

11.1.4 Shocks

In more realistic cases the external force is distributed around a certain frequency Ω, of the order of the frequency ω_0, as given by a gaussian

$$f = const \cdot f_0 \int d\omega_1 e^{-(\omega_1 - \Omega)^2/2\Delta^2} \cos\omega_1 t \ , \tag{11.42}$$

where Δ is the frequency extension of the external spectrum. If $\Delta \ll \Omega$, ω_0 then the external wavepacket is similar with a monochromatic wave, and the results for the amplification factors are similar with those given above. More interesting is the opposite limit $\Delta \gg \Omega$, ω_0, which corresponds to a shock of a short duration as given by

$$f = -f_0 \Delta t e^{-\Delta^2 t^2/2} \ , \tag{11.43}$$

extending over a time interval $t \sim 1/\Delta$. Such a force is obtained as a gaussian distribution of the form

$$f = (f_0/\sqrt{2\pi}\Delta) \int d\omega_1 [-(\omega_1 - \Omega)/\Delta] e^{-(\omega_1 - \Omega)^2/2\Delta^2} \sin\omega_1 t \ . \tag{11.44}$$

Now, it is easy to compute the maximum value of the acceleration under the action of such an external force, $ac_{max} = f_0/m\sqrt{e}$, as well as the velocity $v_{max} = f_0/m\Delta$ and the displacement

$$d = (f_0/m\Delta) \int_{-\infty}^{t} d\tau e^{-\Delta^2\tau^2/2} \ , \tag{11.45}$$

which gives a maximum value $d_{max} = \sqrt{2\pi} f_0/m\Delta^2$.

The coordinate of the particle can easily be obtained by introducing the force f given by equation (11.43) into the general solution of the form given by equation (11.9). We may neglect the small contribution of the damping in this case, and make use of the inequalities $\Delta \gg \Omega$, ω_0 in estimating the integral (11.9). We obtain

$$x = \frac{f_0}{m\Delta} t e^{-\Delta^2 t^2/2} \ , \tag{11.46}$$

which has a maximum value $|x|_{max} = (f_0/m\Delta^2\sqrt{e})$. Therefore, the amplification factor for displacement is

$$F_d = 1/\sqrt{2\pi e} \ . \tag{11.47}$$

Similarly, we get the amplification factors $F_v = 1$ for velocity, and $F_a = 2.28$ for acceleration. As one can see, the amplification is less than unity for displacement, equal to unity for velocity and higher than unity for acceleration for shocks of very short duration.

11.2 Oscillator-wave coupling

In studies of seismic risk and hazard it is important to assess the effects of the seismic motion upon localized structures, either natural or built on Earth's surface. Usually, such structures are viewed as localized harmonic oscillators, with one or several degrees of freedom and corresponding eigenfrequencies (characteristic frequencies). It is assumed that the seismic motion acts as an external force upon such oscillators and the resonant regime is highlighted. It is desirable to avoid the resonance, *i.e.* the structure's characteristic frequencies must be different from the main frequencies of the seismic motion at the site of the structure (local seismic motion).

Two elements are overlooked in such a simplified picture: the reaction of the structure back on the elastic medium and the coupling of the structure to the elastic medium. We show here a way of introducing these two elements in the analysis and describe the consequences of including these two more realistic features.

11.2.1 Structure on the surface

First, we consider the free plane surface of an infinite, homogeneous elastic medium; we consider elastic waves propagating on this surface and assume a generic wave equation

$$\rho \ddot{\mathbf{u}} = F \Delta \mathbf{u} \tag{11.48}$$

describing the motion of the (two-dimensional) displacement vector $\mathbf{u}$; in equation (11.48) ρ is the superficial mass density and F is a generic superficial modulus of elasticity, such that the wave velocity is given by $c^2 = F/\rho$; the laplacian in equation (11.48) is the two-dimensional laplacian. On the other hand we assume a point-like harmonic oscillator with mass m and eigenfrequency Ω localized at $\mathbf{r}_0$ on the surface, described by the equation

$$m\ddot{\mathbf{v}} + m\Omega^2 \mathbf{v} = 0 \ , \tag{11.49}$$

where $\mathbf{v}$ is the oscillator's displacement from its equilibrium position.

The medium acts upon the oscillator with the surface force density $(F\Delta\mathbf{u})_{\mathbf{r}=\mathbf{r}_0}$; for an area S of the contact surface between the oscillator and the medium, the force acting upon the oscillator is $S(F\Delta\mathbf{u})_{\mathbf{r}=\mathbf{r}_0}$. We introduce a coupling function g and write the force as $gS(F\Delta\mathbf{u})_{\mathbf{r}=\mathbf{r}_0}$; under these conditions, the equation of motion of the oscillator becomes

$$m\ddot{\mathbf{v}} + m\Omega^2 \mathbf{v} = gS(F\Delta\mathbf{u})_{\mathbf{r}=\mathbf{r}_0} \ ; \tag{11.50}$$

the area S must be much smaller than the area constructed with any relevant wavelength. The coupling function g may have a complex structure; it may depend on the oscillator eigenmode (frequency Ω), the oscillator amplitude, the local amplitude $\mathbf{u}$ of the wave, or the time t. We assume here the most simple situation which corresponds to a constant g. Obviously, $g \leq 1$.

Similarly, the oscillator reacts back upon the elastic medium, with its inertia force $-gm\ddot{\mathbf{v}}S\delta(\mathbf{r}-\mathbf{r}_0)$, localized at $\mathbf{r}_0$ and affected by the coupling function; the wave equation (11.48) becomes

$$\rho\ddot{\mathbf{u}} = F\Delta\mathbf{u} - gm\ddot{\mathbf{v}}\delta(\mathbf{r}-\mathbf{r}_0) \ . \tag{11.51}$$

Equations (11.50) and (11.51) are two coupled equations. We write equation (11.51) as

$$\frac{1}{c^2}\ddot{\mathbf{u}} - \Delta\mathbf{u} = -\frac{gm\ddot{\mathbf{v}}}{F}\delta(\mathbf{r}-\mathbf{r}_0) \ , \tag{11.52}$$

take the temporal Fourier transform, introduce the modulus of the wavevector $k = \omega/c$ and get

$$\Delta\mathbf{u} + k^2\mathbf{u} = -\frac{gm\omega^2\mathbf{v}}{F}\delta(\mathbf{r}-\mathbf{r}_0) \ ; \tag{11.53}$$

the solution of the equation

$$\Delta u + k^2 u = f \tag{11.54}$$

in two dimensions is given by[2]

$$u = \frac{1}{4i}\int d\mathbf{r}' H_0^{(1)}(k\left|\mathbf{r}-\mathbf{r}'\right|)f(\mathbf{r}') \ , \tag{11.55}$$

since

$$\Delta H_0^{(1)} + k^2 H_0^{(1)} = 4i\delta(\mathbf{r}) \ , \tag{11.56}$$

$H_0^{(1)}$ being the Hankel function of zeroth degree and the first kind; at the same time it is the Green function in equation (11.56); its asymptotic behaviour is given by

$$H_0^{(1)}(kr) \sim \begin{cases} \frac{2i}{\pi}\ln(kr) \ , \ kr \to 0 \ , \\ \sqrt{\frac{2}{\pi kr}}e^{i(kr-\pi/4)} \ , \ kr \to \infty \ . \end{cases} \tag{11.57}$$

Applying these formulae to equation (11.53) we get the particular solution

$$\begin{aligned} \mathbf{u}_p &= -\tfrac{gm\omega^2\mathbf{v}}{4iF}H_0^{(1)}(k\left|\mathbf{r}-\mathbf{r}_0\right|) \simeq \\ &\simeq -\tfrac{gm\omega^2\mathbf{v}}{2\pi F}\ln(k\left|\mathbf{r}-\mathbf{r}_0\right|) \ , \ k\left|\mathbf{r}-\mathbf{r}_0\right| \ll 1 \ ; \end{aligned} \tag{11.58}$$

[2] P. M. Morse and H. Feschbach, *Methods of Theoretical Physics*, vol. 1, McGraw-Hill, NY (1953).

we can see that a localized source generates cylindrical waves on an elastic surface, which have a logarithmic singularity at the source. A solution of the homogeneous equation (11.53) must be added to this particular solution in order to get the general solution; we choose a free wave written as

$$\mathbf{u}_0 = \mathbf{A}\cos\omega_0(t - x/c) \ , \tag{11.59}$$

or its Fourier transform

$$\mathbf{u}_0 = \pi\mathbf{A}\left[\delta(\omega-\omega_0)e^{i\omega_0 x/c} + \delta(\omega+\omega_0)e^{-i\omega_0 x/c}\right] \ . \tag{11.60}$$

Now we compute $(\Delta\mathbf{u})_{\mathbf{r}=\mathbf{r}_0}$, where $\mathbf{u} = \mathbf{u}_p + \mathbf{u}_0$, in order to introduce it in equation (11.50); since $\Delta \ln r_{\mathbf{r}=0} = 2\pi\delta(\mathbf{r})$ we have

$$(\Delta\mathbf{u}_p)_{\mathbf{r}=\mathbf{r}_0} = -\frac{gm\omega^2\mathbf{v}}{F}\delta(\mathbf{r}-\mathbf{r}_0)_{\mathbf{r}=\mathbf{r}_0} \simeq -\frac{gm\omega^2\mathbf{v}}{FS} \ , \tag{11.61}$$

while

$$\begin{gathered}(\Delta\mathbf{u}_0)_{\mathbf{r}=\mathbf{r}_0} = -\tfrac{\omega_0^2}{c^2}\pi\mathbf{A}\cdot \\ \cdot\left[\delta(\omega-\omega_0)e^{i\omega_0 x_0/c} + \delta(\omega+\omega_0)e^{-i\omega_0 x_0/c}\right] \ .\end{gathered} \tag{11.62}$$

Introducing these quantities in the Fourier transform of equation (11.50) we get

$$\begin{gathered}m(\Omega^2-\omega^2)\mathbf{v} = -g^2 m\omega^2\mathbf{v} - \\ -\tfrac{gSF\omega_0^2}{c^2}\pi\mathbf{A}\left[\delta(\omega-\omega_0)e^{i\omega_0 x_0/c} + \delta(\omega+\omega_0)e^{-i\omega_0 x_0/c}\right] \ ,\end{gathered} \tag{11.63}$$

or

$$\begin{gathered}\mathbf{v}(\omega) = -\tfrac{\omega_0^2}{\Omega^2-\omega^2(1-g^2)}\times \\ \times\tfrac{gS\rho}{m}\pi\mathbf{A}\left[\delta(\omega-\omega_0)e^{i\omega_0 x_0/c} + \delta(\omega+\omega_0)e^{-i\omega_0 x_0/c}\right] \ .\end{gathered} \tag{11.64}$$

The most important result exhibited by equation (11.64) is the change in the resonance frequency $\omega \rightarrow \omega\sqrt{1-g^2}$ or $\Omega \rightarrow \Omega/\sqrt{1-g^2}$. As a result of its interaction with the elastic medium, the oscillator eigenfrequncy Ω changes into $\Omega/\sqrt{1-g^2}$ (gets "renormalized").[3] If we take the inverse Fourier transform we get

$$v(t) = -\frac{\omega_0^2}{\Omega^2 - \omega_0^2(1-g^2)} \frac{gS\rho}{m} \mathbf{A} \cos \omega_0 (t - x_0/c) \; ; \tag{11.65}$$

if we take into account the contribution of the poles $\omega_0 = \pm\Omega/\sqrt{1-g^2}$ we get the solution corresponding to free oscillations at resonance, which occurs now at the modified eigenfrequency $\pm\Omega/\sqrt{1-g^2}$. It is worth noting that for a perfect coupling corresponding to $g = 1$, there is not a resonance anymore.

11.2.2 Oscillator in an elastic medium

We examine here the case of a point oscillator embedded in a homogeneous, infinite, elastic medium. Equations (11.50) and (11.51) read now

$$\begin{gathered} m\ddot{\mathbf{v}} + m\Omega^2 \mathbf{v} = gV(F\Delta\mathbf{u})_{\mathbf{R}=\mathbf{R}_0} \; , \\ \rho\ddot{\mathbf{u}} = F\Delta\mathbf{u} - gm\ddot{\mathbf{v}}\delta(\mathbf{R}-\mathbf{R}_0) \; , \end{gathered} \tag{11.66}$$

where ρ is the volume density of mass and F is a generic (volume) modulus of elasticity; the wave velocity has the same expression (and value) given by $c^2 = F/\rho$ and V (much smaller than the volume constructed with any relevant wavelength) is the volume of the oscillator. The wave equation

$$\frac{1}{c^2}\ddot{u} - \Delta u = f \tag{11.67}$$

has the particular solution

$$u = \frac{1}{4\pi} \int d\mathbf{R}' \frac{f(\mathbf{R}', t - |\mathbf{R}-\mathbf{R}'| /c)}{|\mathbf{R}-\mathbf{R}'|} \; ; \tag{11.68}$$

applying this formula to the wave equation

$$\frac{1}{c^2}\ddot{\mathbf{u}} - \Delta\mathbf{u} = -\frac{gm\ddot{\mathbf{v}}}{F}\delta(\mathbf{R}-\mathbf{R}_0) \tag{11.69}$$

[3] B. F. Apostol, "A resonant coupling of a localized harmonic oscillator to an elastic medium", Roum. Reps. Phys. **69** 116 (2017).

we get the particular solution

$$\mathbf{u}_p = -\frac{gm\ddot{\mathbf{v}}(t - |\mathbf{R} - \mathbf{R}_0|/c)}{4\pi F |\mathbf{R} - \mathbf{R}_0|} \tag{11.70}$$

and

$$(\Delta \mathbf{u}_p)_{\mathbf{R}=\mathbf{R}_0} = \frac{gm\ddot{\mathbf{v}}(t)}{F}\delta(\mathbf{R} - \mathbf{R}_0)_{\mathbf{R}=\mathbf{R}_0} = \frac{gm\ddot{\mathbf{v}}(t)}{FV} \ , \tag{11.71}$$

since $\Delta(1/R) = -4\pi\delta(\mathbf{R})$. Similarly, for a plane wave $\mathbf{u}_0 = \mathbf{A}\cos\omega_0(t - x/c)$ we get

$$(\Delta \mathbf{u}_0)_{\mathbf{R}=\mathbf{R}_0} = -(\omega_0^2 \mathbf{A}/c^2)\cos\omega_0(t - x_0/c) \ . \tag{11.72}$$

The equation of motion (11.66) of the oscillator becomes

$$m\ddot{\mathbf{v}} + m\Omega^2\mathbf{v} = g^2 m\ddot{\mathbf{v}} - \frac{gVF\omega_0^2}{c^2}\mathbf{A}\cos\omega_0(t - x_0/c) \ , \tag{11.73}$$

or

$$m(1 - g^2)\ddot{\mathbf{v}} + m\Omega^2\mathbf{v} = -g\rho V\omega_0^2\mathbf{A}\cos\omega_0(t - x_0/c) \ . \tag{11.74}$$

This is a typical equation of motion for a harmonic oscillator with a modified eigenfrequency under the action of an external force.

We may say that the reaction of a point harmonic oscillator to the elastic medium to which it is coupled modifies the inertia of the oscillator, which implies a change in its eigenfrequency; this change is controlled by a coupling function. The introduction of the coupling function and the reaction upon the elastic medium may bring important consequences in estimating the resonance regime of a structure subjected to the action of a seismic motion. The present treatment opens the way of introducing various features in the coupling functions, in order to meet more realistic situations; in particular it is amenable to introducing non-linearities which may affect the coupling of the structure with its site's motion.

11.3 Anharmonic oscillators

There is a huge literature on anharmonic oscillators, both quantum-mechanical and classical.[4] Exact solutions are known for classical cubic and quartic anharmonic oscillators with and without dissipation.[5] We present here a simple derivation of the exact solution for the classical cubic oscillator, the first-order terms in the corresponding series expansion in powers of the anharmonicity and the self-consistent harmonic approximation.

11.3.1 Cubic oscillator

Let $T = m\dot{u}^2/2$ be the kinetic energy and

$$U = \frac{1}{2}m\omega^2 u^2 + \frac{1}{3}m\omega^2 a u^3 \tag{11.75}$$

the potential energy of a cubic anharmonic oscillator of mass m, frequency ω and anharmonicity parameter $a > 0$. The energy conservation gives

$$\dot{u}^2 = \frac{2}{m}(E - U) = \omega^2(x^2 - u^2 - \frac{2}{3}au^3) \ , \tag{11.76}$$

for this oscillator, where $E = m\omega^2 x^2/2 > 0$ is the energy. For $x^2 > 1/3a^2$ the velocity in equation (11.76) vanishes for $u_1 > 0$ and the motion is infinite for $u < u_1$. For $x^2 < 1/3a^2$ the velocity in (11.76) vanishes for $u_3 < u_2 < u_1$ and the motion is infinite for $u < u_3$ and finite for $u_2 < u < u_1$. For this finite motion equation (11.76) can also be written as $\dot{u}^2 = (2a\omega^2/3)(u_1 - u)(u - u_2)(u - u_3)$, and the

[4] L. Skala, J. Cizek, V. Kapsa and E. J. Weniger, "Large-order analysis of the convergent renormalized strong-coupling perturbation theory for the quartic anharmonic oscillator", Phys. Rev. **A56** 4471 (1997); L. Skala, J. Cizek, E. J. Weniger and J. Zarnastil, "Large-order behaviour of the convergent perturbation theory for anharmonic oscillators", Phys. Rev. **A59** 102 (1999).

[5] K. Banerjee, J. K. Bhattacharjee and H. S. Manni, "Classical anharmonic oscillators: rescaling the perturbation series", Phys. Rev. **A30** 1118 (1984); J. M. Dixon, J. A. Tuszynski and M. Otwinowski, "Special analytical solutions of the damped-anharmonic-oscillator equation", Phys. Rev. **A44** 3484 (1991).

integral of motion reads

$$\int_{u_2}^{u} \frac{dy}{\sqrt{(u_1-u)(u-u_2)(u-u_3)}} = \sqrt{2a/3}\omega t \ , \tag{11.77}$$

for $u_2 < u < u_1$ and the initial conditions $u = u_2$, $\dot{u} = 0$ for $t = 0$. The integral in equation (11.77) can be expressed by means of the elliptic function of the first kind $F(\varphi, k)$ by introducing $\sin\alpha = [(u_1 - u_3)(y-u_2)/(u_1-u_2)(y-u_3)]^{1/2}$.[6] We obtain

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} = \tau \ , \tag{11.78}$$

where

$$\sin\varphi = \sqrt{\frac{u_1-u_3}{u_1-u_2}}\sqrt{\frac{u-u_2}{u-u_3}} \ , \tag{11.79}$$

the modulus of the elliptic function is given by

$$k^2 = \frac{u_1-u_2}{u_1-u_3} \ , \tag{11.80}$$

and the dimensionless time τ is given by

$$\tau = \frac{1}{2}\sqrt{u_1-u_3}\sqrt{2a/3}\omega t \ . \tag{11.81}$$

From (11.79) we obtain the solution

$$u = \frac{u_2 - k^2 u_3 \sin^2\varphi}{1-k^2\sin^2\varphi} \ , \tag{11.82}$$

or, making use of the Jacobi sine-amplitude $snF = sn\tau = sin\varphi$,[7] we get

$$u = \frac{u_2 - k^2 u_3 sn^2\tau}{1-k^2 sn^2\tau} \ . \tag{11.83}$$

[6] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products*, Academic Press, London (1980), p. 219, 3.131(5).

[7] I. S. Gradshteyn and I. M. Ryzhik, *loc. cit.*, p. 910.

This is the exact solution of the cubic anharmonic oscillator. It oscillates between u_2 for $\varphi = n\pi$, and u_1 for $\varphi = (2n+1)\pi/2$, n being an integer. The period T of the motion is given by

$$K = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} = \frac{1}{4}\sqrt{2a(u_1-u_3)/3}\omega T \ , \tag{11.84}$$

where K is the complete elliptic function. A similar exact solution can also be obtained for the quartic anharmonic oscillator.[8]

It is worth noting that the infinite motion proceeds in a finite time. Indeed, let $u_1 > 0$ and $u_{2,3} = A \pm iB$ for $x^2 > 1/3a^2$. Then, the integral in equation (11.77) becomes $F(\varphi, k) = \sqrt{2aD/3}\omega t$, where $k^2 = [1 + (u_1 - A)/D]/2$, $D = [(u_1 - A)^2 + B^2]^{1/2}$ and $u = u_1 - D\tan^2(\varphi/2)$. One can see that $u \to -\infty$ for $\varphi \to \pi$, which means that motion goes to infinite in a finite time T_1 given by $2K = \sqrt{2aD/3}\omega T_1$.

It is often useful to have the solution of the cubic oscillator in the limit of the weak anharmonicity. In order to get this limit we need the approximate roots $u_{1,2,3}$ of the equation $x^2 - u^2 - \frac{2}{3}au^3 = 0$ in this limit. Introducing $z = 2au/3$ this equation becomes $z^3 + z^2 - \varepsilon^2 = 0$, where the perturbational parameter is $\varepsilon = 2ax/3$. It is easy now to solve perturbationally this equation; its solutions are given by $z_{1,2} = \pm\varepsilon(1 \mp \varepsilon/2 + \varepsilon^2/4)$ and $z_3 = -1 + \varepsilon^2$, or

$$\begin{gathered} u_1 = x(1-\varepsilon/2+\varepsilon^2/4) \ , \ u_2 = -x(1+\varepsilon/2+\varepsilon^2/4) \ , \\ u_3 = -\tfrac{x}{\varepsilon}(1-\varepsilon^2) \ . \end{gathered} \tag{11.85}$$

Making use of these expansions in powers of ε we obtain $k^2 = 2\varepsilon(1 - \varepsilon + 11\varepsilon^2/4)$ and $K = \pi(1+\varepsilon/2+\varepsilon^2/16)/2$. Using the same expansions in equation (11.84) we get the well-known second-order shift

$$\Omega = 2\pi/T = \omega(1 - 15\varepsilon^2/16) = \omega(1 - 5a^2x^2/12) \tag{11.86}$$

in frequency. Similarly, the angle φ is obtained from equation (11.78) as

$$\varphi = \frac{1}{2}\Omega t + \frac{\varepsilon}{4}\sin\Omega t + \frac{\varepsilon^2}{64}\sin 2\Omega t \ , \tag{11.87}$$

[8] B. F. Apostol, "On anharmonic oscillators", Roum. J. Phys. **50** 915 (2005).

and the oscillator coordinate

$$u = -x\cos\Omega t - \tfrac{x\varepsilon}{4}(3-\cos 2\Omega t) - \\ -\tfrac{x\varepsilon^2}{2}(2-\tfrac{17}{8}\cos\Omega t + 2\cos 2\Omega t - \tfrac{11}{8}\cos 3\Omega t) \ . \quad (11.88)$$

It is worth noting that the renormalized frequency Ω appears in these expansions, instead of the original frequency ω. All these expansions in powers of ε can also be obtained directly by solving perturbationally the equation of motion $\ddot{u} = -\omega^2(u+au^2)$, including the frequency renormalization.

11.3.2 Self-consistent harmonic approximation

Let $T = m\dot{u}^2/2$ be the kinetic energy of a linear oscillator of mass m, and $U = (m\omega^2/2)(u^2 + 2au^3/3)$ its potential energy with cubic anharmonicities, where a is a parameter. Let u^3 be approximated by

$$u^3 = \frac{3}{2}(Au + Bu^2) \ , \quad (11.89)$$

where $A = \bar{u^2}$, $B = \bar{u}$, the averages being taken over the motion and the coefficients 3/2 in equation (11.89) being chosen such as $\bar{u^3} = 3\bar{u}\bar{u^2}$. It is easy to see that the oscillator becomes then a displaced one, with the frequency $\Omega = \omega(1+aB)^{1/2}$; the solution is $u = u_0\cos\Omega t - C$, where u_0 is an amplitude and $C = aA/2(1+aB)$. The condition $\bar{u^3} = 3\bar{u}\bar{u^2}$ is fulfilled only for small values of C, as expected ($\bar{u} = -C$, $\bar{u^2} = u_0^2/2 + C^2$, $\bar{u^3} = -3u_0^2C/2$). It follows $C \cong au_0^2/4$ and $A \cong u_0^2/2$, $B = -C \cong -au_0^2/4$. The frequency shift is then given by

$$\Omega = \omega(1+aB)^{1/2} \simeq \omega(1 - a^2u_0^2/8) \ , \quad (11.90)$$

which compares well with the exact result $\Omega = \omega(1 - 5a^2u_0^2/12)$.

A similar decomposition $u^4 = 3Au^2/2$ holds for the quartic anharmonicity in the potential energy $U = (m\omega^2/2)(u^2 + bu^4/2)$, where $A = \bar{u^2}$ and b is the anharmonic parameter. The condition $\bar{u^4} = 3(\bar{u^2})^2/2$ is then fulfilled exactly ($\bar{u^2} = A = u_0^2/2$, $\bar{u^4} = 3u_0^4/8$ for solution $u = u_0\cos\Omega t$ and frequency $\Omega = \omega(1+3Ab/4)^{1/2}$). It follows the frequency shift given by

$$\Omega = \omega(1+3Ab/4)^{1/2} \cong \omega(1 + 3bu_0^2/16) \ , \quad (11.91)$$

for small b, which again compares well with the exact result $\Omega = \omega(1+3bu_0^2/8)$. Is is worth noting that the frequency shift is quadratic in amplitude for cubic anharmonicities, and linear for quartic anharmonicities.

Similar approximations can be used for higher-order anharmonicities, without any loss of qualitative behaviour, and a satisfactory representation of the quantitative results. They are called generically the self-consistent harmonic approximation.

11.4 Parametric resonance

It is natural to assume that the force $f\cos\omega t$ which acts upon a harmonic oscillator depends on the coordinate u of the oscillator. Then, we may write $f(u) = f_0 + f_1 u + ...$, where $f_0 = f(0)$, $f_1 = f^{'}(0)$, ... and the equation of motion of the oscillator with mass m and eigenfrequency ω_0 becomes

$$\ddot{u} + \omega_0^2\left(1 - f_1/m\omega_0^2\cos\omega t\right)u = (f_0/m)\cos\omega t\ ; \tag{11.92}$$

making use of $h = -f_1/m\omega_0^2$ it is re-written as

$$\ddot{u} + \omega_0^2\left(1 + h\cos\omega t\right)u = (f_0/m)\cos\omega t\ . \tag{11.93}$$

The solution of the homogeneous equation

$$\ddot{u} + \omega_0^2\left(1 + h\cos\omega t\right)u = 0 \tag{11.94}$$

may exhibit a resonance for some particular values of the frequency ω. This is known as the parametric resonance.[9]

Indeed, let us assume $\omega = 2\omega_0$ and $h \ll 1$. In this case equation (11.94) is a Mathieu equation.[10] We seek a solution

$$u \simeq a\cos\omega_0 t + b\sin\omega_0 t\ , \tag{11.95}$$

[9] L. Landau and E. Lifshitz, *Course of Theoretical Physics*, vol. 1., *Mechanics*, Elsevier, Oxford (1976).

[10] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge University Press, Cambridge (1927).

where the coefficients a and b depend slowly on the time t and additional terms with frequencies $\omega_0 \pm 2n\omega_0$, n integer, are neglected. Therefore,

$$\begin{gathered}\dot{u} = \dot{a}\cos\omega_0 t - a\omega_0 \sin\omega_0 t + \\ +\dot{b}\sin\omega_0 t + b\omega_0 \cos\omega_0 t\end{gathered} \tag{11.96}$$

and

$$\begin{gathered}\ddot{u} = -2\dot{a}\omega_0 \sin\omega_0 t + 2\dot{b}\omega_0 \cos\omega_0 t - \\ -a\omega_0^2 \cos\omega_0 t - b\omega_0^2 \sin\omega_0 t\ .\end{gathered} \tag{11.97}$$

Equation (11.94) implies

$$\dot{a} + \frac{h\omega_0}{4}b = 0\ ,\ \dot{b} = \frac{h\omega_0}{4}a = 0\ . \tag{11.98}$$

We can see that the coefficients a and b may increase exponentially in time. The force induces a parametric resonance for $\omega = 2\omega_0$. Actually, the resonance may appear in a narrow range of frequency centered on $2\omega_0$. Similar resonances may appear in much narrower ranges centered on $\omega = 2\omega_0/n$, where n is any integer ($n\omega - \omega_0 = \omega$). Equation (11.94) with $\omega = 2\omega_0$, $4\omega_0$,... is a Hill equation.[11] The range of resonance frequencies is diminished by damping.

[11] E. T. Whittaker and G. N. Watson, *loc. cit.*

12 Surface Waves, Inhomogeneities

12.1 Surface waves. H/V-Ratio

12.1.1 Surface (Rayleigh) waves

As it is well known, the equation of elastic waves in a homogeneous and isotropic body is given by

$$\ddot{\mathbf{u}} = c_t^2 \Delta \mathbf{u} + (c_l^2 - c_t^2) grad\, div \mathbf{u} + \mathbf{F} \tag{12.1}$$

where $\mathbf{u}$ is the local displacement, $c_{t,l}$ are the velocities of sound for transverse and longitudinal waves and $\mathbf{F}$ is an external force (per unit mass). The sound velocities are given by

$$c_t^2 = \frac{E}{2\rho(1+\sigma)} \; , \; c_l^2 = \frac{E(1-\sigma)}{\rho(1+\sigma)(1-2\sigma)} \tag{12.2}$$

where E is Young's modulus, σ is Poison's ratio ($0 < \sigma < 1/2$) and ρ is the density of the body.[1]

We consider a half-space $z < 0$ and an external force

$$\mathbf{F} = -\mathbf{f} e^{-i\Omega t} \delta(\mathbf{r}) e^{\kappa z} \tag{12.3}$$

where $\mathbf{f}$ is a force per unit superficial mass, Ω is the frequency of the force, κ is an attenuation coefficient and $\mathbf{r} = (x, y)$ are in-plane coordinates. This would correspond formally to surface "waves" excited on Earth's surface by seismic waves or other external perturbations.

[1] L. Landau and E. Lifshitz, *Course of Theoretical Physics*, vol. 7, *Theory of Elasticity*, Elsevier, Oxford (1986).

The localization of the force means that we detect the surface waves far away form the source of excitation.

We look for solutions of the form $\mathbf{u} \sim e^{i\mathbf{kr}}e^{\kappa z}$ and introduce the notation $\mathbf{u} = (u_l, u_t, u_v)$ and $\mathbf{k} = (k, 0, 0)$. In addition we assume $\mathbf{f} = (f_l, 0, f_v)$. Equation (12.1) becomes

$$\ddot{u}_l = (-c_l^2k^2 + c_t^2\kappa^2)u_l + i\kappa k(c_l^2 - c_t^2)u_v - f_l e^{-i\Omega t} \ ,$$
$$\ddot{u}_v = (-c_t^2k^2 + c_l^2\kappa^2)u_v + i\kappa k(c_l^2 - c_t^2)u_l - f_v e^{-i\Omega t} \ , \qquad (12.4)$$
$$\ddot{u}_t = c_t^2(-k^2 + \kappa^2)u_t \ .$$

It is easy to see that the homogeneous equations (12.4) have two distinct eigenfrequencies given by $\omega_{l,t}^2 = c_{l,t}^2(k^2 - \kappa^2)$, corresponding to the eigenmodes $u_l \sim ik$, $u_v \sim \kappa$ and $u_l \sim \kappa\,, u_v \sim -ik$.

We introduce the notation $\omega^2 = c_l^2(k^2 - \kappa_l^2) = c_t^2(k^2 - \kappa_t^2)$ and the linear combinations

$$u_l = (ikAe^{\kappa_l z} + \kappa_t Be^{\kappa_t z})e^{ikx} \ ,$$
$$u_v = (\kappa_l Ae^{\kappa_l z} - ikBe^{\kappa_t z})e^{ikx} \qquad (12.5)$$

in order to satisfy the boundary conditions $\sigma_{iz} = 0$ at the free surface $z = 0$, where

$$\sigma_{ij} = \frac{E}{1+\sigma}(u_{ij} + \frac{\sigma}{1-2\sigma}u_{ll}\delta_{ij}) \qquad (12.6)$$

is the stress tensor and $u_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$ is the strain tensor. These are the well-known Rayleigh "waves".[2] From the boundary conditions we get $u_t = 0$ and the equations

$$2i\kappa_l kA + (k^2 + \kappa_t^2)B = 0 \ ,$$
$$\left[\sigma(k^2 + \kappa_l^2) - \kappa_l^2\right] A + i(1-2\sigma)\kappa_t kB = 0 \ . \qquad (12.7)$$

It is easy to see that the ω_l-solution corresponds to $curl\mathbf{u} = 0$ and the ω_t-solution corresponds to $div\mathbf{u} = 0$. Making use of $c_l^2(k^2 - \kappa_l^2) =$

[2]Lord Rayleigh, "On waves propagated along the plane surface of an elastic solid", Proc. London Math. Soc. **17** 4 (1885).

$c_t^2(k^2-\kappa_t^2)=\omega^2$ and of equations (12.2) the second equation (12.7) can also be written as

$$(k^2+\kappa_t^2)A-2i\kappa_t kB=0 \ , \tag{12.8}$$

so that equations (12.7) have solutions providing

$$(k^2+\kappa_t^2)^2=4\kappa_l\kappa_t k^2 \ . \tag{12.9}$$

We introduce the variable ξ defined by $\omega=c_t\xi k$ and, making use of equations (12.2), we get

$$\begin{aligned}\kappa_l^2&=(1-c_t^2\xi^2/c_l^2)k^2=\left[1-\tfrac{1-2\sigma}{2(1-\sigma)}\xi^2\right]k^2 \ ,\\ \kappa_t^2&=(1-\xi^2\ \ )k^2 \ .\end{aligned} \tag{12.10}$$

Equation (12.9) becomes

$$\xi^6-8\xi^4+8\frac{2-\sigma}{1-\sigma}\xi^2-\frac{8}{1-\sigma}=0 \ . \tag{12.11}$$

This equation has a solution close to unity, $\xi\simeq 1$, for $0<\sigma<1/2$. It follows that $\kappa_l\sim k$ and $\kappa_t\sim 0$. The ratio of the two amplitudes is

$$\frac{A}{B}=2i\frac{\sqrt{1-\xi^2}}{2-\xi^2} \ , \tag{12.12}$$

so the amplitude of the κ_l-wave (A) is much smaller than the amplitude of the κ_t-wave (B). The κ_t-wave is a shallow wave with a large penetration depth ($\kappa_t\simeq 0$).

Therefore, for a given frequency ω we have two waves (with ampitudes A and B), both with an in-plane wavevector $\boldsymbol{k}$, related to the frequency by $\omega=c_t\xi k$, and with two fixed attenuation coefficients $\kappa_{l,t}$. Actually, equation (12.11) has two solutions $\pm\xi$, *i.e.* two frequencies $\omega=\pm c_t\xi k$, which means that the displacement goes like $\sim\sin c_t\xi t$, *i.e.* the displacement is in fact a vibration.

12.1.2 H/V-ratio

We pass now to solving equations (12.4) with the force term. The solution is of the form $\mathbf{u}\sim e^{i\mathbf{kr}}e^{\kappa z}e^{-i\Omega t}$, where κ is the attenuation coefficient in the force. We get easily

$$u_l = \frac{(\Omega^2 - c_t^2 k^2 + c_l^2 \kappa^2) f_l - i\kappa k (c_l^2 - c_t^2) f_v}{\Delta} \ ,$$
$$u_v = \frac{(\Omega^2 - c_l^2 k^2 + c_t^2 \kappa^2) f_v - i\kappa k (c_l^2 - c_t^2) f_l}{\Delta} \qquad (12.13)$$

where $\Delta = \left[\Omega^2 - c_l^2(k^2 - \kappa^2)\right]\left[\Omega^2 - c_t^2(k^2 - \kappa^2)\right]$. We define the "$H/V$"-ratio as $H/V = |u_l|^2 / |u_v|^2$.[3] It is convenient to introduce the notation $s = f_l^2/f_v^2$. We get

$$H/V = \frac{(\Omega^2 - c_t^2 k^2 + c_l^2 \kappa^2)^2 s + \kappa^2 k^2 (c_l^2 - c_t^2)^2}{(\Omega^2 - c_l^2 k^2 + c_t^2 \kappa^2)^2 + \kappa^2 k^2 (c_l^2 - c_t^2)^2 s} \ . \qquad (12.14)$$

We introduce $\omega = c_t \xi k$ for $\xi \simeq 1$ and $r = c_l/c_t$. Let us assume, as a working hypothesis, $\kappa \simeq 0$, by analogy with κ_t given by $\kappa_t^2 = (1-\xi^2)k^2$ for $\xi \simeq 1$. Equation (12.14 becomes then approximately

$$H/V \simeq \frac{(\Omega^2 - \omega^2)^2 s}{\left(\Omega^2 - r^2\omega^2\right)^2} \ . \qquad (12.15)$$

We can see that the H/V-ratio exhibits a resonance at $\omega = \omega_0 \simeq \Omega/r = (c_t/c_l)\Omega$.[4] If we take $\Omega = c_l k$ this resonance is in the vicinity of the S-wave frequency $\omega \simeq c_t k$. For $s = 0$ the H/V-ratio is given by

$$H/V \simeq \frac{(1-\xi^2)(r^2-1)^2\omega^4}{\left(\Omega^2 - r^2\omega^2\right)^2} \qquad (12.16)$$

and one can see that the resonance is rather sharp. For $s \to \infty$ the resonance disappears. A plot of the ratio H/V given by equation (12.15) is shown in Fig. 12.1

[3] M. Nogoshi and T. Igarashi, "On the amplitude characteristics of microtremor-Part 2", J. Seism. Soc. Japan **24** 26 (1971) (in Japanese); Y. Nakamura, "A method for dynamical characteristics estimation of subsurface using microtremor on the ground surface", Quart. Rep. Railway Tech. Res. Inst. **30** 25 (1989); "On the H/V spectrum", The 14th World Conference on Earthquake Engineering, 2008, Beijing, China.

[4] B. F. Apostol, "Resonance of the surface waves. The H/V ratio", Roum. Reps. Phys. **60** 91 (2008).

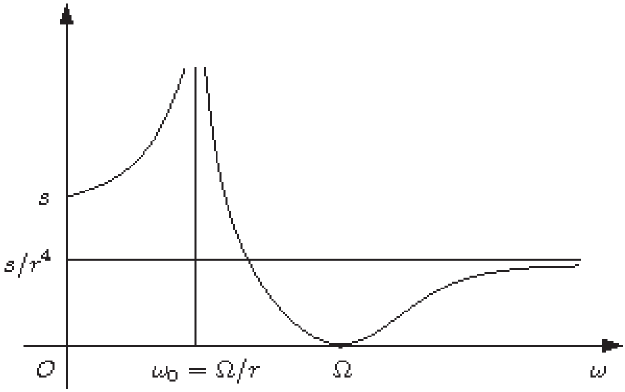

Figure 12.1: The H/V resonance at frequency $\omega_0 = \Omega/r$.

We may also use $\kappa^2 = (1-\xi^2/r^2)k^2$, by analogy with κ_l, and equation (12.14) becomes

$$H/V \simeq \frac{\left[\Omega^2 + (r^2-2)\omega^2\right]^2 s + (r^2-1)^3\omega^4/r^2}{\left[\Omega^2 - (r^2+1/r^2-1)\omega^2\right]^2 + s(r^2-1)^3\omega^4/r^2} \ . \qquad (12.17)$$

This expression has a rather broad maximum. For $s = 0$ equation (12.17) exhibits a resonance at $\omega \simeq (r^2 + 1/r^2 - 1)^{-1/2}\Omega = (1 + 1/r^4 - 1/r^2)^{-1/2}\omega_0$ which is greater than ω_0 ($r^2 > 2$). For $s \to \infty$ the maximum of equation (12.17) disappears.

The surface waves given by (12.13) and the H/V-ratio (12.14) acquire a very simple expression. A small but finite value of κ shifts the resonance frequency and smooth out the resonance, giving it a small width. If the force is a superposition of various frequencies Ω then the resonance is smoothed out and gets a finite width.

It is worth noting that what we call herein surface waves are not the Rayleigh surface waves. If they were, the frequency Ω of the force would be equal to the frequency ω and the H/V-resonance would disappear. We use the term surface waves in the sense that they are proportional to $e^{\kappa z}$and κ may be very small. It is difficult to ascertain an external force proportional to $e^{\kappa z}$. Boundary forces exist, generated by displacement at the surface, but they are restricted to

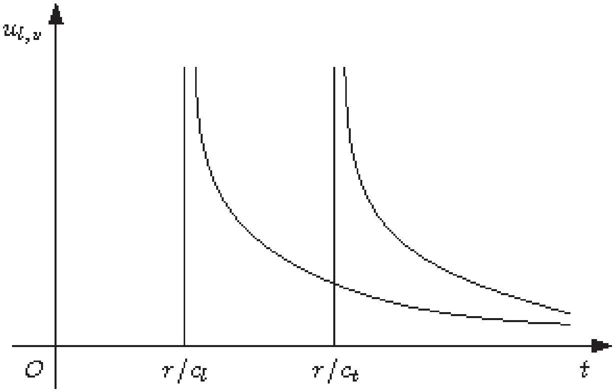

Figure 12.2: The wave fronts of the excited surface waves.

the surface.[5]

12.1.3 Surface displacement

For a very small attenuation coefficient κ the surface waves given by equation (12.13) become

$$u_l \simeq \frac{f_l}{\Omega^2 - c_l^2 k^2} \ , \ u_v \simeq \frac{f_v}{\Omega^2 - c_t^2 k^2} \ . \tag{12.18}$$

It is worth computing the (inverse) Fourier transforms of the displacements given by equations (12.18).[6] We assume that the external force is smoothly distributed with an average $\overline{f}_{l,v}$ over a large range $\Delta\omega$ of frequencies, *i.e.* $1/\Delta\omega$ is approximately the duration of the external pulse. For vanishing κ we may omit the z-dependence. Then, the displacements as function of position and time are given approximately by

$$u_{l,v}(\mathbf{r}, t) \simeq \frac{\overline{f}_{l,v}}{(2\pi)^3 \Delta\Omega} \int d\mathbf{k} \int d\Omega \frac{1}{\Omega^2 - c_{l,t}^2 k^2} e^{-i\Omega t} e^{i\mathbf{kr}} \ . \tag{12.19}$$

[5]B. F. Apostol, *Seismology*, Nova, NY (2020).
[6]B. F. Apostol, "Surface waves in an isotropic semi-infinite body", Roum. Reps. Phys. **65** 1204 (2013).

The integration over Ω can be performed immediately. Similarly, the integration over angles (in $e^{i\mathbf{kr}}$) gives the Bessel function $J_0(kr)$ of the first kind and zeroth order. We get

$$u_{l,v}(\mathbf{r},t) \simeq -\frac{\overline{f}_{l,v}}{4\pi c_{l,t}\Delta\Omega}\int_0^\infty dk \cdot J_0(kr)\sin c_{l,t}kt \ . \qquad (12.20)$$

The remaining integral in equation (12.20) is a known integral.[7] We get

$$u_{l,v}(\mathbf{r},t) \simeq -\frac{\overline{f}_{l,v}}{4\pi c_{l,t}\Delta\Omega}\frac{\theta(c_{l,t}t-r)}{\sqrt{c_{l,t}^2t^2-r^2}} \ , \qquad (12.21)$$

where $\theta(x)=1$ for $x>1$ and $\theta(x)=0$ for $x<0$ is the step function. This is a cylindrical wave, with an abrupt wavefront at $r=c_{l,t}t$, propagating with the velocities $c_{l,t}$ and decreasing in time as $\sim 1/t$. Its duration is of the order of the pulse duration $1/\Delta\Omega$.[8] An illustration of these waves as functions of time is given in Fig. 12.2. A finite value of κ smooths out the abrupt wavefront of these waves.

We might say that surface waves excited by an external force in an isotropic half-space may exhibit a resonance of their H/V-ratio (horizontal to vertical polarization), which is rather sharp and it may be close to the frequency of the in-plane transverse (shear) waves. The dominant soil frequencies for a response caused by such an external force are the resonance frequency $\omega_0=\Omega/r$. It may be close to the frequency c_tk of the in-plane (shear) transverse wave (for $\Omega=c_lk$), or to the frequency c_t^2k/c_l for $\Omega=c_tk$ and vanishing longitudinal force ($f_l=0$). Similarly, if we assume that Ω is the fundamental frequency of a superposed layer of thickness d ($\Omega=\pi c_t^0/2d$, where c_t^0 is the transverse wave velocity in the layer), the resonance frequencies are $\pi c_tc_t^0/2c_ld$. These formulae may help in estimating the thickness of the layer or the magnitude of velocities. If we recall that we have denoted here the resonance frequency by $c_tk=2\pi c_t/\lambda$, we get $\lambda/d=4c_l/c_t^0$. If the assumption $\lambda/d=4$ is correct, this would imply $c_l/c_t^0=1$. If,

---

[7] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, NY (2000), p. 709, 6.671.

[8] H. Lamb, "On wave-propagation in two dimensions", Proc. London Math. Soc. **35** 141 (1902); "On the propagation of tremors over the surface of an elastic solid," Phil. Trans. Roy. Soc. (London) **A203** 1 (1904).

on the other side, we assume that the resonance frequency is given by $\Omega/r^0 = c_t^0\Omega/c_l^0 = \pi c_t^0/2d$, then, for $\Omega = r\omega_0 = 2\pi r c_t/\lambda$, we get $\lambda/d = 4c_l/c_l^0$; for $\lambda/d = 4$ we have $c_l/c_l^0 = 1$.

The disturbance given by equation (12.21) is similar to the main shock. We recall that the main shock goes like $(c_{l,t}^2 t^2 - r^2)^{-3/2}$, so the disturbance given by equation (12.21) is less singular than the main shock, while its tail is higher than the tail of the main shock. Since this disturbance arises from a force concentrated at $\boldsymbol{r} = 0$, it is likely that it is produced by the epicentral region, at distances much larger than the dimension of this region; the epicentral region is left aside in estimating the main shock.[9]

Also, we note that the horizontal (radial) displacement both for the main shock and the epicentral shock is propagated with velocity c_l, while the vertical displacement in both cases is propagated with velocity $c_t < c_l$. Therefore, at the moment a large (singular) horizontal displacement is measured the vertical one is zero, or, at most, it is the finite vertical displacement in the preceding S (or P) wave, a circumstance which gives indeed a large H/V-ratio.

It is worth noting that a large H/V-ratio seems to occur only on a soft soil, superposed over a hard bedrock, so it may be related to the oscillation frequency of the superposed layer.[10] On the other hand, the main shock which appears in the presence of an internal discontinuity[11] leads to a H/V-ratio of the order $r/2z_1$ for long times (low frequencies), where r is the epicentral distance and z_1 is the depth of the discontinuity. This contribution arises from the waves propagating with longitudinal velocity.

12.1.4 An exponentially decaying force

We consider a semi-infinite isotropic elastic solid (half-space) extending over the region $z < 0$. We are interested in its elastic motion

[9] The time Fourier transform of the function in equation (12.21) is related to the integral $\int_0^{+\infty} du e^{ix\cosh u} = \frac{i\pi}{2} H_0^{(1)}(x)$, where $H_0^{(1)}$ is the Hankel function (I. S. Gradshteyn and i. M. Ryzhik, *Table of Integrals, Series and Products*, Academic, NY (2000), p. 904, 8.421).

[10] Y. Nakamura, "On the H/V spectrum", The 14th World Conference on Earthquake Engineering, 2008, Beijing, China.

[11] B. F. Apostol, *Seismology*, Nova, NY (2020).

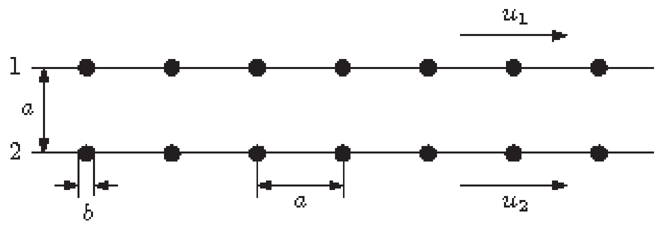

Figure 12.3: A schematic representation of two parallel threads of atoms.

(elastic waves) under the action of an external force, in particular an external damping force of the form $f \sim e^{\kappa z}$, where κ is an attenuation (damping) coefficient. We sketch here an argument that such a force is plausible within the general definition of an elastic solid.

First, we resort to the general atomic structure of solid matter. In a simplified model we may focus attention upon two parallel threads of atoms, arranged at their equilibrium positions in a generic solid, as shown in Fig. 12.3. We recall that one of the basic assumptions of the linear elasticity consists of admitting only local displacements which, though they may be large, must have a small spatial derivative, *i.e.* they must vary very slowly within the solid. This means that atoms in thread 1 in Fig. 12.3 may move only within the inter-atomic spaces of thread 2 (each such space is of length a in Fig. 12.3). If we look to a large stack of such many parallel threads, it is easy to see indeed that the displacements may get high values, but certainly they vary slowly over large distances. This amounts to disregard the friction, or other losses in an ideal elastic solid.

For external forces the situation may be different, especially near their location, where we may expect abrupt changes in atomic positions. Suppose that such a force f acts along the thread 1 and causes a displacement u_1 of this thread. Atoms in thread 1 may pass over atoms in thread 2. As long as atoms in thread 1 move within the inter-atomic space of thread 2 we may expect little displacement of thread 2. Each time, however, when atoms in thread 1 pass over atoms in thread 2 we may expect a certain contribution to the displacement u_2 of thread 2. It is reasonable to assume that such a displacement is of the order of the atomic size denoted by b in Fig. 12.3. Suppose that there are n such pass-over steps. We get approximately a displacement

$$u_2 = nb \ . \tag{12.22}$$

Now, the number n is obviously proportional to u_1/a, so we have

$$u_2 = \frac{b}{a}u_1 \ . \tag{12.23}$$

Suppose now that we take a stack of parallel chains, the first, denoted by u_0 located at $z = 0$ and the n-th, denoted by u_n located at z. Obviously, we have

$$u_n = \left(\frac{b}{a}\right)^n u_0 \tag{12.24}$$

by iterating equation (12.23), or

$$u(z) = u_0 e^{-\frac{z}{a}\ln(b/a)} = u_0 e^{\kappa z} \ , \tag{12.25}$$

where $\kappa = -\frac{1}{a}\ln(b/a)$; we have taken into account that $z < 0$, $\ln(b/a) < 0$ and have assumed the same distance a between the threads. Such a damping displacement, caused by an external force, may generate an elastic force which, obviously, has the same damping character: $f \sim e^{\kappa z}$ (through $f_i = \partial\sigma_{ij}/\partial x_j$, where σ_{ij} is the stress tensor). This is, of course the external force. It may even have a abrupt decrease (though not necessarily), for large κ, but, of course, this does not impede on the basic assumption of the elasticity, because its magnitude may be sufficiently small as to preserve the small variations of the elastic displacement.

12.1.5 Rough surface

We consider the elastic waves equation

$$\ddot{\mathbf{u}} = c_t^2\Delta\mathbf{u} + (c_l^2 - c_t^2) grad\, div\mathbf{u} \ , \tag{12.26}$$

where $\mathbf{u}$ is the displacement field,

$$c_t = \sqrt{E/2\rho(1+\sigma)} \ , \ c_l = \sqrt{E(1-\sigma)/\rho(1+\sigma)(1-2\sigma)} \tag{12.27}$$

are the velocities of the transverse and longitudinal waves, E is the Young's modulus, σ is the Poisson ratio ($0 < \sigma < 1/2$) and ρ is the density of the isotropic body. As it is well known, the elastic field can be written as $\mathbf{u} = \mathbf{u}_t + \mathbf{u}_l$, with $div\mathbf{u}_t = 0$ and $curl\mathbf{u}_l = 0$. This splitting leads to $\ddot{\mathbf{u}}_t = c_t^2 \Delta \mathbf{u}_t$ and $\ddot{\mathbf{u}}_l = c_l^2 \Delta \mathbf{u}_l$. We consider surface waves of the form $e^{-i\omega t} e^{i\mathbf{kr}} e^{\kappa z}$ which are vanishing for $z \to -\infty$. The frequencies are given by $\omega_t^2 = c_t^2(k^2 - \kappa_t^2)$ and $\omega_l^2 = c_l^2(k^2 - \kappa_l^2)$. The above conditions of transversality and irotationality lead to the following representation for these waves:

$$\begin{aligned} \mathbf{u}_l &= A(k_x, k_y, -i\kappa_l) e^{-i\omega t} e^{i\mathbf{kr}} e^{\kappa_l z} \ , \\ \mathbf{u}_t &= (B\kappa_t, C\kappa_t, -i(Bk_x + Ck_y)) e^{-i\omega t} e^{i\mathbf{kr}} e^{\kappa_t z} \ , \end{aligned} \tag{12.28}$$

where A, B, C are coefficients which depend on the wavevector $\mathbf{k}$ (for a given frequency). The elastic field is a superposition over $\mathbf{k}'s$ of $\mathbf{u}_l + \mathbf{u}_t$ given by equation (12.28).

We consider a free surface with a normal vector $\mathbf{n}$ and impose the free-force boundary condition $\sigma_{ij} n_j = 0$, where

$$\sigma_{ij} = \frac{E}{1+\sigma}(u_{ij} + \frac{\sigma}{1-2\sigma} u_{ll} \delta_{ij}) \tag{12.29}$$

is the stress tensor and $u_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2$ is the strain tensor. We give here the strain tensor as computed from equation (12.28):

$$\begin{aligned} u_{xx} &= \sum_{\mathbf{k}} i \left(k_x^2 A_{\mathbf{k}} e^{\kappa_l z} + k_x \kappa_t B_{\mathbf{k}} e^{\kappa_t z} \right) e^{i\mathbf{kr}} \ , \\ u_{yy} &= \sum_{\mathbf{k}} i \left(k_y^2 A_{\mathbf{k}} e^{\kappa_l z} + k_y \kappa_t C_{\mathbf{k}} e^{\kappa_t z} \right) e^{i\mathbf{kr}} \ , \\ u_{zz} &= -\sum_{\mathbf{k}} i \left[\kappa_l^2 A_{\mathbf{k}} e^{\kappa_l z} + \kappa_t \left(k_x B_{\mathbf{k}} + k_y C_{\mathbf{k}} \right) e^{\kappa_t z} \right] e^{i\mathbf{kr}} \end{aligned} \tag{12.30}$$

and

$$u_{xy} = \sum_{\mathbf{k}} i\left[k_x k_y A_{\mathbf{k}} e^{\kappa_l z} + \frac{1}{2}\kappa_t \left(k_y B_{\mathbf{k}} + k_x C_{\mathbf{k}}\right) e^{\kappa_t z}\right] e^{i\mathbf{kr}} \ ,$$

$$u_{xz} = \sum_{\mathbf{k}} [\kappa_l k_x A_{\mathbf{k}} e^{\kappa_l z} +$$

$$+\frac{1}{2}\left((k_x^2 + \kappa_t^2)B_{\mathbf{k}} + k_x k_y C_{\mathbf{k}}\right) e^{\kappa_t z}]e^{i\mathbf{kr}} \ , \tag{12.31}$$

$$u_{yz} = \sum_{\mathbf{k}} [\kappa_l k_y A_{\mathbf{k}} e^{\kappa_l z} +$$

$$+\frac{1}{2}\left(k_x k_y B_{\mathbf{k}} + (k_y^2 + \kappa_t^2)C_{\mathbf{k}}\right) e^{\kappa_t z}]e^{i\mathbf{kr}} \ .$$

In addition

$$u_{ll} = \sum_{\mathbf{k}} i(k^2 - \kappa_l^2) A_{\mathbf{k}} e^{\kappa_l z} e^{i\mathbf{kr}} \ . \tag{12.32}$$

We consider the free surface described by equation $z = f(x, y)$. Its normal vector $\mathbf{n}$ is proportional to $(f_1, f_2, -1)$, where $f_1 = \partial f/\partial x$ and $f_2 = \partial f/\partial y$. We assume that f_1 and f_2 are small quantities and limit ourselves to the second-order expansion in these quantities; we get the normal vector $\mathbf{n} = (f_1, f_2, -(1 - f_1^2/2 - f_2^2/2))$. The boundary condition $\sigma_{ij} n_j = 0$ reads

$$(u_{xx} + \tfrac{\sigma}{1-2\sigma} u_{ll}) f_1 + u_{xy} f_2 = u_{xz}(1 - f_1^2/2 - f_2^2/2) \ ,$$

$$u_{xy} f_1 + (u_{yy} + \tfrac{\sigma}{1-2\sigma} u_{ll}) f_2 = u_{yz}(1 - f_1^2/2 - f_2^2/2) \ , \tag{12.33}$$

$$u_{xz} f_1 + u_{yz} f_2 = (u_{zz} + \tfrac{\sigma}{1-2\sigma} u_{ll})(1 - f_1^2/2 - f_2^2/2) \ .$$

We assume further that $\kappa_{l,t} f \ll 1$, and write $e^{\kappa_{l,t} z} \simeq 1 + \kappa_{l,t} f + \kappa_{l,t}^2 f^2/2$. We use the Fourier decomposition $f = \sum_{\mathbf{q}} f_{\mathbf{q}} e^{i\mathbf{qr}}$. The system of equations (12.33) can then be solved, in principle. It is of the form $A_{ij} X_j = \lambda_{ij} X_j$, where λ denotes a small parameter, originating in function f. For a flat surface $\lambda = 0$ ($z = 0$) and we get the Rayleigh waves. For a surface defined by function f the above system leads to $det(A - \lambda) = 0$ which induces small changes in the $\mathbf{k}$-dependence of $\kappa_{l,t}$; and leads also to small changes in coefficients A, B, C. Essentially, the effect of the irregular surface is to introduce a superposition of waves instead of a pure $\mathbf{k}$-Rayleigh plane wave. The actual superposition depends on the particular form of the function f.

We adopt a statistical view and average over function f and its derivatives in equations (12.33). We assume $\bar{f}_{1,2} = 0$, $\overline{ff}_{1,2} = 0$, $\bar{f^2} = 2f^2$ and $\overline{f_{1,2}^2} = q^2 f^2$, where f is now a parameter and q^{-1} is a measure of the scale of the variations of the function f. Then, the wavevectors are not coupled anymore in equations (12.33), so we may restrict ourselves to one **k**-plane wave; in addition, we may assume $k_y = 0$. The *lhs* of equations (12.33) is now vanishing, and the remaining *rhs* gives

$$
\begin{aligned}
&2\kappa_l k A(1+\kappa_l^2 f^2 - q^2 f^2)+ \\
&+(k^2+\kappa_t^2)B(1+\kappa_t^2 f^2 - q^2 f^2) = 0 \ , \\
&\kappa_l^2 A(1+\kappa_l^2 f^2 - q^2 f^2) - \kappa_t k B(1+\kappa_t^2 f^2 - q^2 f^2) - \\
&-\tfrac{\sigma}{1-2\sigma}(k^2-\kappa_l^2)A(1+\kappa_l^2 f^2 - q^2 f^2) = 0
\end{aligned}
\tag{12.34}
$$

and $C = 0$. We introduce new variables $A' = A(1+\kappa_l^2 f^2 - q^2 f^2)$ and $B' = B(1+\kappa_t^2 f^2 - q^2 f^2)$ and the above equations become

$$
\begin{aligned}
&2\kappa_l k A' + (k^2+\kappa_t^2)B' = 0 \ , \\
&2\kappa_t k B' + (k^2+\kappa_t^2)A' = 0 \ .
\end{aligned}
\tag{12.35}
$$

The second equation (12.35) is obtained by using the dispersion relations (frequency ω *vs* wavevector **k**) and the definition of the waves velocities as functions of σ.

Equations (12.35) define the Rayleigh waves. With $\omega = c_l k\xi$ and $\kappa_l^2 = (1 - c_t^2\xi^2/c_l^2)k^2$, $\kappa_t^2 = (1-\xi^2)k^2$ they lead to

$$
\xi^6 - 8\xi^4 + 8\frac{2-\sigma}{1-\sigma}\xi^2 - \frac{8}{1-\sigma} = 0 \ , \tag{12.36}
$$

which has a solution close to unity. The amplitude ratio is given by

$$
A'/B' = -2\frac{\sqrt{1-\xi^2}}{2-\xi^2} \ . \tag{12.37}
$$

The statistical effect of small superficial irregularities on the surface waves is the small change in amplitudes according to $A' = A(1+\kappa_l^2 f^2 - q^2 f^2)$ and $B' = B(1+\kappa_t^2 f^2 - q^2 f^2)$.

12.2 Surface inhomogeneities

12.2.1 Introduction

The effect of the surface inhomogeneities (defects) on the propagation of the elastic waves in a semi-infinite isotropic solid body (half-space) is investigated herein. A perturbation-theoretical scheme is devised for small surface defects (in comparison with the relevant elastic disturbances propagating in the body), and the elastic waves equations are solved in the first-order approximation. It is shown that surface defects generate both scattered waves localized (and propagating only) on the surface (two-dimensional waves) and scattered waves reflected back in the body. Directional effects, wave slowness and attenuation by diffusive scattering, or possible resonance effects are discussed.

There is a great deal of interest in the role played by the surface defects (inhomogeneities) in a large variety of physical phenomena, ranging from mechanical properties of the elastic bodies, to hydrodynamical flow of microfluids, dispersive properties of surface plasmon-polariton in nanoplasmonics, terahertz-waves generation or electronic microstructures. Giant corrugations have been found on the graphite surface by scanning tunneling microscopy, due to the elastic deformations induced by atomic forces between tip and surface. Periodic surface corrugation plays a central role in enhanced, or suppressed, optical transmission in the subwavelength regime, or in highly-directional optical emission. An appreciable reduction in the thermal conductance has been assigned to the phonon scattering by surface defects. Stick-slip instability responsible for earthquakes has been studied, as well as the associated radiation of seismic surface waves. It has been recognized that elastic-waves propagation effects may play a central role in the surface defects associated with the cracks occurring in heterogeneous media, like aluminium alloys, ceramics or rock. The main difficulty in getting more definite results in this problem resides in modelling conveniently the surface inhomogeneities, such as to arrive at mathematically operational approaches.

We introduce here a model of surface inhomogeneities, whose elastic characteristics are, in general, distinct from the ones of the underlying (isotropic) elastic half-space (semi-infinite solid).[12] Such a model may account both for surface roughness and for surface coatings (in general, non-uniform). It is shown that the elastic waves propagating in the semi-infinite body (incident on and reflected specularly by the surface) generate a force localized on the surface, which is responsible for the scattered waves. This force arises mainly from both the presence of the surface layer and the more-or-less abrupt termination of the solid at its surface. The scattered waves are of two kinds: localized (and propagating only) on the surface (two-dimensional waves), and waves scattered back in the body. For an enhanced distribution of surface defects the waves scattered back in the body may get confined to the surface (damped surface waves). The method employed herein is based on a perturbation-theoretical scheme, and the resulting coupled integral equations are solved in the first approximation with respect to the magnitude of the defects distribution. Multiple scattering is expected to occur in higher-order approximations. The perturbation method employed here differs from other perturbation methods. For instance, the perturbation treated here is partially an intrinsic one, not a purely external one, as in the Born approximation. The introduction of a surface layer is equivalent to some extent with a double-scale treatment, so that, in this respect, there is a resemblance with a multi-scale method.

Forward scattering and backward scattering of elastic waves have also been reported in corrugated waveguides. Great insight has been obtained previously in the coupling of the surface (Rayleigh) waves to periodic corrugation (grating), especially as regards the wave attenuation, slowing and leaking (outgoing increasing wave), corroborated with band gaps and stop bands, by using non-perturbational techniques. The reflection and refraction of elastic (acoustic) waves at a rough surface, or ducts with variable cross-sections, have been extensively studied, emphasizing the role of the boundary conditions. Powerful numerical methods have been developed for such complex problems. A great deal of attention was given to the coupled modes propagating in elastic waveguides with rough surfaces, which highlighted a rich phenomenology. The interplay between mode dispersion

[12] B. F. Apostol, "The effect of surface inhomogeneities on the propagation of elastic waves", J. Elas. **114** 85 (2014).

and surface roughness may lead to a well-defined selectivity in the transmission coefficient and anomalous backscattering enhancement. Most of such important results are obtained numerically. Similar results have been reported for sound and ultrasound waves propagating in fluids.

In addition to such results, we show here that the surface inhomogeneities may cause localized waves, propagating only on the surface, which may store a certain amount of energy, due to the localization effects. Attenuation of crustal waves across the Alpine range has been reported, which might be associated with the localization of energy in the surface defects region.[13] The method presented here can be extended to electromagnetic waves,[14] or fluid waves, propagating in a semi-infinite body with surface defects. It was employed to analyze the elastic waves produced by localized forces in semi-infinite solids.[15]

12.2.2 Elastic body with surface inhomogeneities

We consider an isotropic elastic body extended boundlessly along the directions $\mathbf{r} = (x, y)$ and limited along the z-direction by a free surface $z = h(\mathbf{r})$, where $h(\mathbf{r}) > 0$ is a function to be further specified (roughness function). The body, which may also be termed a semi-infinite solid (elastic half-space) with a non-planar surface, occupies the region $z < h(\mathbf{r})$. It is convenient to write the well-known equation for free elastic waves in an isotropic body as

$$\frac{1}{v_t^2}\ddot{\mathbf{u}} - \Delta\mathbf{u} = m \cdot grad\, div\mathbf{u} \ , \tag{12.38}$$

where $\mathbf{u}(\mathbf{r},\, z,\, t)$ is the displacement field, t denotes the time, v_t is the velocity of the transverse waves, $m = v_l^2/v_t^2 - 1 > 1/3$ (actually 1)[16] and v_l is the velocity of the longitudinal waves. Indeed, equation

[13] M. Campillo, B. Feignier, M. Bouchon and N. Bethoux, "Attenuation of crustal waves across the Alpine range", J. Geophys. Res. **98** 1987 (1993).

[14] B. F. Apostol, "Scattering of electromagnetic waves from a rough surface", J. Mod. Optics **59** 1607 (2012).

[15] B. F. Apostol, "Elastic waves in a semi-infinite body", Phys. Lett. **A374** 1601 (2010).

[16] L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics*, vol. 7, *Theory of Elasticity*, Elsevier, Oxford (2002).

(12.38) gives the free transverse waves ($div\mathbf{u} = 0$) propagating with velocity v_t and the free longitudinal waves ($curl\mathbf{u} = 0$) propagating with velocity v_l.

For a semi-infinite body with a surface described by equation $z = h(\mathbf{r})$ and extending in the region $z < h(\mathbf{r})$ the displacement field can be written as

$$\mathbf{u} = (\mathbf{v},\, w)\theta[h(\mathbf{r}) - z] \ , \tag{12.39}$$

where $\mathbf{v}$ lies in the (x, y)-plane, w is directed along the z-axis and θ is the step function ($\theta(z) = 0$ for $z < 0$, $\theta(z) = 1$ for $z > 0$). The magnitude of the surface inhomogeneities (deviation from a plane) is given by the function $h(\mathbf{r})$, which we assume to be very small in comparison with the relevant wavelengths along the z-directions of the elastic disturbances propagating in the body. Consequently, we may use the first-order approximation

$$\mathbf{u} = (\mathbf{v},\, w)[\theta(-z) + h(\mathbf{r})\delta(z)] \tag{12.40}$$

for equation (12.39), where $\delta(z)$ is the Dirac function. This is the usual approximation employed in the perturbation-theoretical approaches.[17] The specific conditions of validity for this approximation will be discussed on the final results.

We write such a displacement field as

$$\mathbf{u} = \mathbf{u}_0 + \delta\mathbf{u}_0 \ , \tag{12.41}$$

where

$$\mathbf{u}_0 = (\mathbf{v}_0,\, w_0)\theta(-z)\,,\ \ \delta\mathbf{u}_0 = \left.(\mathbf{v}_0,\, w_0)\right|_{z=0} h\delta(z) \ , \tag{12.42}$$

and assume that $\mathbf{u}_0$ satisfies the wave equation (12.38)

$$\frac{1}{v_t^2}\ddot{\mathbf{u}}_0 - \Delta\mathbf{u}_0 = m \cdot grad\, div\mathbf{u}_0 \tag{12.43}$$

[17] F. Gilbert and L. Knopoff, "Seismic scattering from topographic irregularities", J. Geophys. Res. **65** 3437 (1960); J. A. Ogilvy, "Wave scattering from rough surfaces", Reps. Progr. Phys. **50** 1553 (1987) and references therein; A. A. Maradudin and D. L. Mills, "The attenuation of Rayleigh surface waves by surface roughness", Ann. Phys. **100** 262 (1976).

with specific boundary conditions at $z = 0$. This equation describes incident and (specularly) reflected waves propagating in a semi-infinite solid with a plane surface $z = 0$. We can see that $\delta\mathbf{u}_0$ generates a source-term localized on the surface (a force), which can produce scattered waves. We denote the displacement field associated with these scattered waves by $\mathbf{u}_1$; it satisfies the wave equation

$$\frac{1}{v_t^2}\ddot{\mathbf{u}}_1 - \Delta\mathbf{u}_1 = m \cdot grad\, div\mathbf{u}_1 + \frac{\mathbf{f}}{v_t^2} \ , \tag{12.44}$$

where the force is given by

$$\frac{\mathbf{f}}{v_t^2} = \frac{1}{v_t^2}\delta\ddot{\mathbf{u}}_0 - \Delta\delta\mathbf{u}_0 - m \cdot grad\, div\delta\mathbf{u}_0 \ . \tag{12.45}$$

Equations (12.44) and (12.45) represent merely a different way of re-writing the wave equation for a semi-infinite solid with surface defects. For waves localized on the surface the solution of equation (12.44) is $\mathbf{u}_1 = \delta\mathbf{u}_0$. Another solutions are given by the waves scattered back in the body by the surface defects*, i.e.* waves generated in equation (12.44) by the source term $\mathbf{f}$ (a particular solution of equation (12.44)). We generalize this model of surface defects by assuming that the roughness is "inhomogeneous", *i.e.* it is a homogeneous elastic medium with different elastic characteristics than the plane-surface half-space bulk (for instance, different density and elastic constants). Therefore, we introduce distinct velocities $\overline{v}_{t,l}$ and denote all the changed parameters with an overbar (for instance, $\overline{m} = \overline{v}_l^2/\overline{v}_t^2 - 1$). The force is given in this case by

$$\frac{\overline{\mathbf{f}}}{v_t^2} = \frac{1}{\overline{v}_t^2}\delta\ddot{\mathbf{u}}_0 - \Delta\delta\mathbf{u}_0 - \overline{m} \cdot grad\, div\delta\mathbf{u}_0 \ , \tag{12.46}$$

The results are expressed conveniently by using the relative differences $\eta_{t,l} = 1 - v_{t,l}^2/\overline{v}_{t,l}^2$. The displacement field $\mathbf{u}_1$ given by equation (12.44) can be written as $\mathbf{u}_1 = (\mathbf{v},\, w)\theta(-z)$.

We may say that, in the presence of a displacement field $\mathbf{u}_0$, the surface inhomogeneities generate a force $\overline{\mathbf{f}}$, localized on the surface and of the same order of magnitude as the function h ($\delta u_0 \sim h\delta(z)$). This force is the difference between the inertial force $\delta\ddot{\mathbf{u}}_0/\overline{v}_t^2$ and the elastic force $\Delta\delta\mathbf{u}_0 + \overline{m} \cdot grad \cdot div\delta\mathbf{u}_0$; it represents the distinct way the surface follows the elastic motion in comparison with the bulk. Equation (12.43) gives the free incident and reflected waves propagating in a half-space with a plane surface, while equation (12.44) gives the scattered waves produced by the roughness of the surface, as a consequence of the source term $\overline{\mathbf{f}}/v_t^2$.

It is worth noting that such a model of inhomogeneous surface may correspond either to a surface whose physical properties have been changed, or to a solid which is homogeneous everywhere, including its rough surface. Indeed, in the latter case, it is precisely the spatial variations of the rough surface which affect its elastic properties, viewed as a homogeneous medium, and render it, in fact, a rough surface which is inhomogeneous with respect to the bulk.

The above perturbation-theoretical scheme can also be writen in a different way, by recasting equation (12.38) into an equation involving the velocity v_l of the longitudinal waves and the parameter $n = 1 - v_t^2/v_l^2 = m/(1+m)$. Then, equations (12.43) - (12.45) become

$$\begin{aligned} &\tfrac{1}{v_l^2}\ddot{\mathbf{u}}_0 - \Delta\mathbf{u}_0 = n(-\Delta\mathbf{u}_0 + grad\, div\mathbf{u}_0) \ , \\ &\tfrac{1}{v_l^2}\ddot{\mathbf{u}}_1 - \Delta\mathbf{u}_1 = n(-\Delta\mathbf{u}_1 + grad\, div\mathbf{u}_1) + \tfrac{\overline{\mathbf{f}}}{v_l^2} \ , \end{aligned} \tag{12.47}$$

where

$$\frac{\overline{\mathbf{f}}}{v_l^2} = \frac{1}{\overline{v}_l^2}\delta\ddot{\mathbf{u}}_0 - (1-\overline{n})\Delta\delta\mathbf{u}_0 - \overline{n} \cdot grad\, div\delta\mathbf{u}_0 \ . \tag{12.48}$$

We solve equation (12.44) and the second equation (12.47) for the scattered transverse and longitudinal waves by using the Green function method.

12.2.3 Plane surface

As it is well known, the elementary solutions of equation (12.43), or the first equation (12.47), (homogeneous elastic waves equation) for a half-space with a plane surface are transverse and longitudinal plane waves of the form

$$\mathbf{u}_0 \sim (e^{\pm i\kappa_0 z}, e^{\pm i\kappa_0' z})e^{-i\omega t + i\mathbf{k}_0\mathbf{r}} \ , \tag{12.49}$$

where both incident ($+\kappa_0$, $+\kappa_0^{'}$) and reflected ($-\kappa_0$, $-\kappa_0^{'}$) waves are included, ω is the frequency and $\mathbf{k}_0$ is the in-plane wavevector. For $div\mathbf{u}_0 = 0$ we get the transverse waves, propagating with the velocity v_t ($\omega = v_t K_0$, where $\mathbf{K}_0 = (\mathbf{k}_0, \kappa_0)$), with the z-component of the wavevector $\kappa_0 = \sqrt{\omega^2/v_t^2 - k_0^2}$. For $curl\mathbf{u}_0 = 0$ we get the longitudinal waves (through $curl \cdot curl\mathbf{u}_0 = -\Delta\mathbf{u}_0 + grad \cdot div\mathbf{u}_0 = 0$), propagating with the velocity v_l and the z-component of the wavevector $\kappa_0^{'} = \sqrt{\omega^2/v_l^2 - k_0^2}$ ($\omega = v_l K_0^{'}$ and $\mathbf{K}_0^{'} = (\mathbf{k}_0, \kappa_0^{'})$). The transverse waves have two polarizations, one in the propagating plane (the ($\mathbf{k}_0$, κ_0)-plane), which we call here the p-wave (parallel wave), another perpendicular to the propagating plane, which we call here the s-wave (from the German "senkrecht", which means "perpendicular"). Linear combinations of the plane waves given by equation (12.49) are subject to conditions imposed on the surface (*e.g.*, free or fixed surface). The p- and s- notations are used in electromagnetism. In seismology the longitudinal waves, denoted here by the suffix l, are usually called primary waves and denoted by P, while the transverse s-waves discussed here are called shear horizontal waves and denoted by SH. The transverse p-waves discussed here have not a simple polarization with respect to the surface. It is worth noting that the results of the perturbation scheme applied here to the integral equations acquire their most simple and convenient form for longitudinal waves and p- and s-transverse waves as used here.

We derive here these free waves propagating in a half-space with a plane surface by a different method, which will be used subsequently in deriving the solutions for the scattered waves (equation (12.44) and the second equation (12.47)). In order to simplify the notations we omit here the subscript 0.

The solution of equation (12.43) is written as

$$\mathbf{u} = [\mathbf{v}(z),\, w(z)]\,\theta(-z)e^{-i\omega t + i\mathbf{kr}} \ . \tag{12.50}$$

Introducing this $\mathbf{u}$ in equation (12.43) and leaving aside the exponential factor $e^{-i\omega t + i\mathbf{kr}}$ we get

$$\frac{\partial^2 \mathbf{u}}{\partial z^2} + \kappa^2 \mathbf{u} = \mathbf{S} \ , \tag{12.51}$$

where $\kappa^2 = \omega^2/v_t^2 - k^2$ and the source $\mathbf{S}$ has the components

$$\begin{gathered}
\mathbf{S}_{(x,y)} = -im\mathbf{k}\left(i\mathbf{kv} + \tfrac{\partial w}{\partial z}\right)\theta(-z) + \\
+\left(\tfrac{\partial \mathbf{v}}{\partial z}\big|_{z=0} + imk\, w|_{z=0}\right)\delta(z) + \mathbf{v}|_{z=0}\,\delta^{'}(z) \ , \\
S_z = -m\left[i\mathbf{k}\tfrac{\partial \mathbf{v}}{\partial z} + \tfrac{\partial^2 w}{\partial z^2}\right]\theta(-z) + im\,\mathbf{kv}|_{z=0}\,\delta(z) + \\
+(1+m)\left[\tfrac{\partial w}{\partial z}\big|_{z=0}\delta(z) + w|_{z=0}\,\delta^{'}(z)\right] \ .
\end{gathered} \tag{12.52}$$

We can see that the source $\mathbf{S}$, which collects all the contributions from $m \cdot grad\mathbf{u}$ and the derivatives of $\theta(-z)$ in $\Delta\mathbf{u}$, acts as an "external force" in equation (12.51). As it is well known, the particular solution of equation (12.51) is given by

$$\mathbf{u}(z) = \int dz' G(z - z')\mathbf{S}(z') \ , \tag{12.53}$$

where

$$G(z) = \frac{1}{2i\kappa} e^{i\kappa|z|} \tag{12.54}$$

is the Green function for equation (12.51) (Green function of the one-dimensional Helmholtz equation). Making use of the notations $v_1 = \mathbf{vk}/k$ and $v_2 = \mathbf{vk}_\perp/k$, where $\mathbf{k}_\perp$is a vector perpendicular to $\mathbf{k}$ and of the same magnitude k, equations (12.52)-(12.54) lead to

$$v_2 = -\frac{i}{2\kappa}\left.\frac{\partial v_2}{\partial z}\right|_{z=0} e^{-i\kappa z} - \frac{1}{2}\, v_2|_{z=0}\, e^{-i\kappa z} \tag{12.55}$$

and

$$\begin{array}{c} v_1 = -\frac{imk^2}{2\kappa}\int^0 dz' v_1(z')e^{i\kappa|z-z'|}- \\ -\frac{mk}{2\kappa}\frac{\partial}{\partial z}\int^0 dz' w(z')e^{i\kappa|z-z'|}- \\ -\frac{i}{2\kappa}\left.\frac{\partial v_1}{\partial z}\right|_{z=0} e^{-i\kappa z} - \frac{1}{2}\left. v_1\right|_{z=0} e^{-i\kappa z} \ , \\ (1+m)w = -\frac{mk}{2\kappa}\frac{\partial}{\partial z}\int^0 dz' v_1(z')e^{i\kappa|z-z'|}+ \\ +\frac{im\kappa}{2}\int^0 dz' w(z')e^{i\kappa|z-z'|}- \\ -\frac{i}{2\kappa}\left.\frac{\partial w}{\partial z}\right|_{z=0} e^{-i\kappa z} - \frac{1}{2}\left. w\right|_{z=0} e^{-i\kappa z} \ . \end{array} \tag{12.56}$$

Equation (12.55) corresponds to the s-wave. It is easy to see that the particular solution given by equation (12.55) is identically vanishing. Therefore, we are left with the free s-waves given by equation (12.49), as expected ($\sim e^{\pm i\kappa z}e^{-i\omega t+i\mathbf{kr}}$).

Let us take the second derivative of equations (12.56) with respect to z and use the identity

$$\frac{\partial^2}{\partial z^2}\int dz' f(z')e^{i\kappa|z-z'|} = -\kappa^2\int dz' f(z')e^{i\kappa|z-z'|} + 2i\kappa f(z) \tag{12.57}$$

for any arbitrary function $f(z)$. We get

$$\begin{array}{c} \frac{\partial^2 v_1}{\partial z^2} + \kappa^2 v_1 = -imk\left(ikv_1 + \frac{\partial w}{\partial z}\right) \ , \\ \frac{\partial^2 w}{\partial z^2} + \kappa^2 w = -m\frac{\partial}{\partial z}\left(ikv_1 + \frac{\partial w}{\partial z}\right) \ . \end{array} \tag{12.58}$$

We can see that for $div(v_1,\, w) = 0$, *i.e.* for $ikv_1 + \partial w/\partial z = 0$, we get the free p-waves ($\kappa = \sqrt{\omega^2/v_t^2 - k^2}$), according to equation (12.49) ($\sim e^{\pm i\kappa z}e^{-i\omega t+i\mathbf{kr}}$). Similarly, for $curl\mathbf{u} = 0$, *i.e.* for $ikw - \partial v_1/\partial z = 0$, equations (12.58) become

$$(1+m)\frac{\partial^2(v_1,\, w)}{\partial z^2} + (\kappa^2 - mk^2)(v_1,\, w) = 0 \ , \tag{12.59}$$

or, making use of $m = v_l^2/v_t^2 - 1$,

$$\frac{\partial^2(v_1,\, w)}{\partial z^2} + \kappa'^2(v_1,\, w) = 0 \ , \tag{12.60}$$

where $\kappa' = \sqrt{\omega^2/v_l^2 - k^2}$, *i.e.* free longitudinal waves $\sim e^{\pm i\kappa' z} e^{-i\omega t + i\mathbf{kr}}$.

The longitudinal waves can also be obtained by noticing that the coupled equations (12.56) imply the relationship

$$\frac{\partial v_1}{\partial z} - ikw = Ce^{-i\kappa z} \ , \tag{12.61}$$

where

$$C = -\frac{1}{2}\left(\frac{\partial v_1}{\partial z} - ikw\right)\Big|_{z=0} + \frac{1}{2}\left(i\kappa v_1 - \frac{k}{\kappa}\frac{\partial w}{\partial z}\right)\Big|_{z=0} \ . \tag{12.62}$$

We use this relationship in one of equations (12.58), and get

$$\frac{\partial^2 v_1}{\partial z^2} + \kappa'^2 v_1 = -\frac{im\kappa}{1+m} Ce^{-i\kappa z} \ . \tag{12.63}$$

The particular solution of this equation is vanishing identically, and we are left with free longitudinal waves. Indeed, equation (12.61) with $C = 0$ corresponds to $curl(v_1, w) = 0$.

The p-waves are obtained in a similar way, by starting with the first equation (12.47). Using $\mathbf{u}$ given by an equation similar with equation (12.50) we get

$$\begin{gathered}(1-n)v_2 = \tfrac{in(\kappa'^2+k^2)}{2\kappa'} \int^0 dz' v_2(z') e^{i\kappa' |z-z'|} - \\ -\tfrac{i}{2\kappa'} \left.\tfrac{\partial v_2}{\partial z}\right|_{z=0} e^{-i\kappa' z} - \tfrac{1}{2} \left. v_2\right|_{z=0} e^{-i\kappa' z}\end{gathered} \tag{12.64}$$

and

$$(1-n)v_1 = \tfrac{in\kappa'}{2}\int^0 dz' v_1(z')e^{i\kappa'|z-z'|}-$$
$$-\tfrac{nk}{2\kappa'}\tfrac{\partial}{\partial z}\int^0 dz' w(z')e^{i\kappa'|z-z'|}-$$
$$-\tfrac{i}{2\kappa'}\left.\tfrac{\partial v_1}{\partial z}\right|_{z=0} e^{-i\kappa' z} - \tfrac{1}{2}\left.v_1\right|_{z=0} e^{-i\kappa' z} \ ,$$
$$w = -\tfrac{nk}{2\kappa'}\tfrac{\partial}{\partial z}\int^0 dz' v_1(z')e^{i\kappa'|z-z'|}+ \tag{12.65}$$
$$+\tfrac{ink^2}{2\kappa'}\int^0 dz' w(z')e^{i\kappa|z-z'|}-$$
$$-\tfrac{i}{2\kappa'}\left.\tfrac{\partial w}{\partial z}\right|_{z=0} e^{-i\kappa' z} - \tfrac{1}{2}\left.w\right|_{z=0} e^{-i\kappa' z} \ .$$

It is easy to see, by taking the second derivative with respect to z, that equation (12.64) gives the free s-waves. Similarly, by taking the second derivative with respect to z, equations (12.65) become

$$\tfrac{\partial^2 v_1}{\partial z^2} + \tfrac{\kappa'^2}{1-n}v_1 = -ink\tfrac{\partial w}{\partial z} \ ,$$
$$\tfrac{\partial^2 w}{\partial z^2} + (1-n)\kappa^2 w = -ink\tfrac{\partial v_1}{\partial z} \tag{12.66}$$

(where we have used the identity $\kappa'^2 + nk^2 = (1-n)\kappa^2$). On the other hand, from equations (12.65), we get easily the relationship

$$\frac{\partial v_1}{\partial z} + i\frac{\kappa^2}{k}w = \frac{C'}{1-n}e^{-i\kappa' z} \ , \tag{12.67}$$

where

$$C' = -\frac{1}{2}\left(\frac{\partial v_1}{\partial z} + \frac{i\kappa'^2}{k}w\right)\Bigg|_{z=0} + \frac{1}{2}\left(i\kappa' v_1 + \frac{\kappa'}{k}\frac{\partial w}{\partial z}\right)\Bigg|_{z=0} \ . \tag{12.68}$$

Making use of this relationship in equations (12.66) we get

$$\frac{\partial^2 w}{\partial z^2} + \kappa^2 w = -\frac{ink}{1-n}C' e^{-i\kappa' z} \tag{12.69}$$

and a similar equation for v_1. It is easy to see that the particular solution of equation (12.69) is identically vanishing, so we are left with the free p-waves. Indeed, equation (12.67) with $C' = 0$ corresponds to $div(v_1, w) = 0$.

12.2.4 Scattered waves

We consider now a bulk incident transverse wave and reflected transverse and longitudinal waves given by

$$\mathbf{u}_0 = \left(\mathbf{u}_0^{(1)} e^{i\kappa_0 z} + \mathbf{u}_0^{(2)} e^{-i\kappa_0 z} + \mathbf{u}_0^{(3)} e^{-i\kappa_0' z}\right) e^{-i\omega t + i\mathbf{k}_0\mathbf{r}} \tag{12.70}$$

(for $z < 0$), where the amplitudes $\mathbf{u}_0^{(1,2,3)}$ satisfy the corresponding conditions of transverse and longitudinal waves. For instance, in the representation $\mathbf{u}_0 = (\mathbf{v}_0, w_0)$ we have $\mathbf{k}_0\mathbf{v}_0^{(1,2)} \pm \kappa_0 w_0^{(1,2)} = 0$ (including $w_0^{(1,2)} = 0$ for the s-waves) and $\kappa_0\mathbf{v}_0^{(3)}\mathbf{k}_0/k_0 + k_0 w_0^{(3)} = 0$. In addition, the wave given by equation (12.70) must satisfy the conditions at the surface. For instance, for a fixed surface we have $\mathbf{u}_0|_{z=0} = 0$, while for a free surface, we impose the condition $\sigma_{iz} = 0$, where σ_{ij} is the stress tensor ($i = x,\, y,\, z$). All these conditions fix the amplitudes $\mathbf{u}_0^{(1,2,3)}$, up to the incidence angle and the amplitude of the incident wave, in terms of the reflection coefficients and reflection angles, ultimately in terms of the wave velocities $v_{t,l}$. For an incident s-wave we have only a reflected s-wave ($\mathbf{u}_0^{(3)} = 0$), while for an incident p-wave we have both p- and longitudinal waves. A similar situation occurs for an incident longitudinal wave, with κ_0 and κ_0' interchanged in equation (12.70). The displacement $\delta\mathbf{u}_0$ given by equation (12.42) implies $\mathbf{u}_0$ for $z = 0$, so that we may represent this localized contribution of the $\mathbf{u}_0$-wave as

$$\mathbf{u}_0|_{z=0} = (\mathbf{v}_0,\, w_0) e^{-i\omega t + i\mathbf{k}_0\mathbf{r}} \ , \tag{12.71}$$

where $\mathbf{v}_0$, w_0 include contributions corresponding to various polarizations.

First, we are interested in solving equation (12.44) for the scattered waves, with the force $\overline{\mathbf{f}}/v_t^2$ generated by the free waves $\mathbf{u}_0$, as given by equation (12.46). We consider a Fourier component of the form

$$h(\mathbf{r}) = h e^{i\mathbf{q}\mathbf{r}} \tag{12.72}$$

for the roughness function, where h is an amplitude (depending on $\mathbf{q}$) and $\mathbf{q}$ denotes a characteristic wavevector (in final results the contribution $\mathbf{q} \to -\mathbf{q}$ must be included). The localized displacement $\delta\mathbf{u}_0$ given by equation (12.42) can be written as

$$\delta\mathbf{u}_0 = h(\mathbf{v}_0,\, w_0) e^{-i\omega t + i\mathbf{k}\mathbf{r}} \delta(z) \ , \tag{12.73}$$

where $\mathbf{k} = \mathbf{k}_0 + \mathbf{q}$. Making use of this displacement $\delta\mathbf{u}_0$, the force $\overline{\mathbf{f}}/v_t^2$ given by equation (12.46) can be computed straightforwardly. Leaving aside the exponential factor $e^{-i\omega t+i\mathbf{kr}}$, it is given by

$$\begin{aligned}\frac{\overline{\mathbf{f}}_{(x,y)}}{v_t^2} &= -h[\overline{\kappa}^2\mathbf{v}_0\delta(z) + \mathbf{v}_0\delta^{''}(z) - \\ &-\overline{m}\mathbf{k}(\mathbf{k}\mathbf{v}_0)\delta(z) + i\overline{m}\mathbf{k}w_0\delta^{'}(z)] \ , \\ \frac{\overline{f}_z}{v_t^2} &= -h[\overline{\kappa}^2 w_0\delta(z) + w_0\delta^{''}(z) + \\ &+i\overline{m}\mathbf{k}\mathbf{v}_0\delta^{'}(z) + \overline{m}w_0\delta^{''}(z)] \ ,\end{aligned} \tag{12.74}$$

where

$$\overline{\kappa} = \sqrt{\omega^2/\overline{v}_t^2 - k^2} \tag{12.75}$$

and

$$\kappa = \sqrt{\omega^2/v_t^2 - k^2} = \sqrt{\kappa_0^2 - 2\mathbf{k}_0\mathbf{q} - q^2} \ . \tag{12.76}$$

We add the contributions arising from this force (via the Green function of equation (12.51)) to the *rhs* of equations (12.55) and (12.56) and solve these equations by the procedure described in the previous section. For instance, equation (12.55) becomes

$$\begin{aligned}v_2 = &-\frac{i}{2\kappa}\left.\frac{\partial v_2}{\partial z}\right|_{z=0} e^{-i\kappa z} - \frac{1}{2}\left. v_2\right|_{z=0} e^{-i\kappa z} - \\ &-\frac{ih}{2\kappa}(\overline{\kappa}^2 - \kappa^2)v_{02}e^{-i\kappa z} + hv_{02}\delta(z) \ .\end{aligned} \tag{12.77}$$

The displacement v_2 given above includes the localized wave

$$v_{2l} = hv_{02}\delta(z)e^{-i\omega t+i\mathbf{kr}} \ , \tag{12.78}$$

which is a scattered wave propagating only on the surface (two-dimensional wave). The remaining contribution to equation (12.77) (terms without $\delta(z)$) represents scattered waves reflected back in the body. We denote this contribution by v_{2r}. Taking the second derivative with respect to z in equation (12.77) and using the self-consistency condition imposed by this equation on the displacement on the surface, we get immediately the solution

$$v_{2r} = -\frac{ih}{4\kappa}(\overline{\kappa}^2 - \kappa^2)v_{02}e^{-i\omega t+i\mathbf{kr}-i\kappa z} \ . \tag{12.79}$$

This is an s-wave, scattered back in the body by the surface roughness. We can see that it is the distinct elastic parameters of the surface roughness that ensure this scattering (through $\overline{\kappa}^2 - \kappa^2 = -\omega^2\eta_t/v_t^2 \neq 0$). The occurrence of the wavevector $\mathbf{k} = \mathbf{k}_0 + \mathbf{q}$ in equation (12.79) is indicative of the selective reflection phenomenon, associated with corrugated surfaces, and in general, of directional effects.

In likewise manner we get the equations for v_1 and w with the force terms given by equation (12.74). We get the amplitudes for localized waves

$$v_{1l} = hv_{01}\delta(z) \ , \ w_l = h\frac{1+\overline{m}}{1+m}w_0\delta(z) \ . \qquad (12.80)$$

Equations (12.58) and (12.61) remain the same, but the constant C given by equation (12.62) (entering the relationship (12.61)) becomes now

$$C = -\tfrac{1}{2}\left(\tfrac{\partial v_1}{\partial z} - ikw\right)\Big|_{z=0} + \tfrac{1}{2}\left(i\kappa v_1 - \tfrac{k}{\kappa}\tfrac{\partial w}{\partial z}\right)\Big|_{z=0} - \\ -\tfrac{h}{2\kappa}(\overline{\kappa}^2 - \kappa^2)(\kappa v_{01} + kw_0) \ . \qquad (12.81)$$

Following the same procedure as described in the previous section we get the scattered waves

$$v_{1r} = -ih\tfrac{v_t^2}{4\omega^2}(\overline{\kappa}^2 - \kappa^2)(\kappa v_{01} + kw_0)e^{-i\omega t + i\mathbf{kr} - i\kappa z} = \\ = \tfrac{i}{4}h\eta_t(\kappa v_{01} + kw_0)e^{-i\omega t + i\mathbf{kr} - i\kappa z} \qquad (12.82)$$

and $w_r = kv_{1r}/\kappa$. We can see that this represent a p-wave ($div(v_{1r}, w_r) = 0$, *i.e.* $kv_{1r} - \kappa w_r = 0$).

We turn now to the second equation (12.47) with the force given by

$$\frac{\overline{\mathbf{f}}_{(x,y)}}{v_l^2} = -h[(1-\overline{n})\overline{\kappa}^2\mathbf{v}_0\delta(z) + (1-\overline{n})\mathbf{v}_0\delta''(z) - \\ -\overline{n}\mathbf{k}(\mathbf{kv}_0)\delta(z) + i\overline{n}\mathbf{k}w_0\delta'(z)] \ , \\ \frac{\overline{f}_z}{v_l^2} = -h[(1-\overline{n})\overline{\kappa}^2 w_0\delta(z) + (1-\overline{n})w_0\delta''(z) + \\ +i\overline{n}\mathbf{kv}_0\delta'(z) + \overline{n}w_0\delta''(z)] \ . \qquad (12.83)$$

By using the procedure described in the previous section we get a localized displacement

$$\mathbf{v}_l = h\frac{1-\overline{n}}{1-n}\mathbf{v}_0\delta(z)\ ,\ w_l = hw_0\delta(z)\ . \tag{12.84}$$

We can see, by comparing equations (12.78), (12.80) and (12.84) that the inhomogeneous roughness affects the localized waves in different ways. For the scattered waves reflected back in the body, equations (12.66) and (12.67) from the previous section remain unchanged, but the constant $C^{'}$ given by equation (12.68) (entering the relationship (12.67)) becomes

$$\begin{aligned} C^{'} = &-\tfrac{1}{2}\left(\tfrac{\partial v_1}{\partial z} + \tfrac{i\kappa^{'2}}{k}w\right)\Big|_{z=0} + \tfrac{1}{2}\left(i\kappa^{'}v_1 + \tfrac{\kappa^{'}}{k}\tfrac{\partial w}{\partial z}\right)\Big|_{z=0} - \\ &-\tfrac{h}{2k}(\overline{\kappa'}^2 - \kappa'^2)(kv_{01} - \kappa' w_0)\ . \end{aligned} \tag{12.85}$$

We get straightforwardly the reflected waves

$$\begin{aligned} v_{1r} = &-ih\tfrac{v_l^2 k}{4\omega^2\kappa'}(\overline{\kappa'}^2 - \kappa'^2)(kv_{01} - \kappa' w_0)e^{-i\omega t + i\mathbf{kr} - i\kappa' z} = \\ &= \tfrac{i}{4}h\eta_l(kv_{01} - \kappa' w_0)e^{-i\omega t + i\mathbf{kr} - i\kappa' z} \end{aligned} \tag{12.86}$$

and $w_r = -\kappa' v_{1r}/k$. We can see that this scattered wave is a longitudinal wave ($curl(v_{1r}, w_r) = 0$, *i.e.* $-\kappa' v_{1r} = kw_r$).

According to equations (12.79), (12.82) and (12.86), within the present model of surface roughness we get waves scattered back in the body only for a rough surface with elastic characteristics different from those of the body (inhomogeneous roughness, $\eta_{t,l} \neq 0$). For a homogeneous roughness, *i.e.* for $\eta_{t,l} = 0$, we get only scattered waves localized on the surface, given by

$$\mathbf{u}_l = \delta\mathbf{u}_0 = h(\mathbf{r})(\mathbf{v}_0, w_0)e^{-i\omega t + i\mathbf{k}_0\mathbf{r}}\delta(z)\ , \tag{12.87}$$

as expected.

12.2.5 Discussion

The localized waves have the general form of the incoming wave $e^{-i\omega t + i\mathbf{k}_0\mathbf{r}}$, modulated by the roughness function $h(\mathbf{r})$. If $\mathbf{q}$ is a characteristic wavevector of this roughness function and $\mathbf{k} = \mathbf{k}_0 + \mathbf{q}$, the velocity of the localized waves is given by $v_s = \omega/k = v_{t,l}k_0/k\sin\theta$, where θ is the incidence angle of the incoming (transverse or longitudinal) wave. The directional effects are clearly seen from the presence of $k = \sqrt{k_0^2 + 2\mathbf{k}_0\mathbf{q} + q^2}$ in the denominator of this relation. It is worth noting that for $\mathbf{q} = \pm\mathbf{k}_0$, *i.e.* for a surface distribution of defects modulated with the same wavelength as the original $\mathbf{u}_0$-wave, there appear scattered waves with half the wavelength of the original $\mathbf{u}_0$-waves (wavevector $2\mathbf{k}_0$) and the whole surface suffers a vibration (independent of the coordinate $\mathbf{r}$), a characteristic resonance phenomenon ($\mathbf{k} = 0$). The waves corresponding to the wavevector $2\mathbf{k}_0$ have a velocity $\omega/2k_0$, which is twice as small as the original velocity on the surface. This is indicative of the slowness phenomenon, associated with rough surfaces.

The $\mathbf{q} = \pm\mathbf{k}_0$ resonance phenomenon is exhibited also by the waves scattered back in the body. Another resonance phenomenon may appear for $\pm 2\mathbf{k}_0\mathbf{q} + q^2 = 0$, which is the well-known Laue-Bragg condition for the X-rays diffraction in crystalline bodies.[18] In this case, $k = k_0$, $\kappa = \kappa_0$ and $\kappa^{'} = \kappa_0^{'}$, and we can see that the scattered transverse (longitudinal) waves are generated only by the transverse (longitudinal) part in the original $\mathbf{u}_0$-waves, as expected, due to the presence of the factors $\kappa v_{01} + k w_0$ and $k v_{01} - \kappa' w_0$ in equations (12.82) and (12.86). For antiparallel $\mathbf{k}_0$ and $\mathbf{q}$ the scattered wave propagates in opposite direction with respect to the incident wave.

The results given above hold also for purely imaginary values of the wavevectors κ or $\kappa^{'}$, when the scattered waves become confined to the surface (surface waves), a situation which may occur especially for high values of the magnitude q of the characteristic wavevectors $\mathbf{q}$ ($q \gg k_0$). According to equations (12.79), (12.82) and (12.86), the scattered waves are now damped ($\sim e^{|\kappa|z}$, $\sim e^{|\kappa'|z}$) and their amplitudes are proportional to the roughness function $h(\mathbf{r})$. It is worth noting that these surface waves are generated by the surface roughness.

As it is well known, the energy of the incident wave is transferred to the reflected waves. In the present case, it is transferred both to the reflected waves as well as to the scattered waves, including the waves localized on the surface and the waves scattered back in the body.

[18] C. Kittel, *Introduction to Solid State Physics*, Wiley, NJ (2005).

According to equations (12.79), (12.82) and (12.86) the energy density of the scattered waves reflected back in the body is proportional to $(h/\overline{\lambda})^2$, where $\overline{\lambda}$ is a characteristic "wavelength" of these waves (projection of the wavelength λ on the surface, or on the direction perpendicular to the surface, or combinations of these). It follows that the validity criterion for our perturbation-theoretical scheme is $h \ll \overline{\lambda}$. In the limit of small roughness ($h \rightarrow 0$), the energy of the scattering waves (their amplitude) is vanishing. It is worth estimating the energy of the waves localized on the surface. For simplicity, we consider a homogeneous roughness, with the localized waves given by

$$(\mathbf{v}_l,\, w_l) = h(\mathbf{v}_0,\, w_0)\delta(z)e^{-i\omega t+i\mathbf{kr}} \tag{12.88}$$

(according to equation (12.87)) and choose the wavevector $\mathbf{k}$ directed along the x-axis. The validity condition for these waves is obtained by assuming that the distribution of the surface defects extends over a distance of the order of $h_m = \max h(\mathbf{r})$ and use the representation $\delta(z) \simeq 1/h_m$ for the δ-function. Then, the perturbation calculations are valid for $\overline{h} \ll h_m$, where $\overline{h}$ is the average (mean value) of the function $h(\mathbf{r})$. This means that the surface distribution of defects has only a few spikes. As it is well known, the (elastic) energy density (per unit mass) can be expressed as

$$\mathcal{E}/\rho = v_t^2(u_{ij}^2 - u_{ii}^2) + \frac{1}{2}v_l^2 u_{ii}^2 \ , \tag{12.89}$$

where $u_{ij} = (1/2)(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$ is the strain tensor. In our case, we use for computing this strain tensor the displacement given by equation (12.88). The strain tensor includes factors proportional to $\delta(z)$ and $\delta^{'}(z)$, and the energy density includes factors proportional to $\delta^2(z)$ and $\delta^{'2}(z)$. The leading contribution comes from $\delta^{'2}(z)$ -terms:

$$\mathcal{E}/\rho = \frac{h^2}{2}(v_t^2\mathbf{v}_0^2 + v_l^2 w_0^2)\delta^{'2}(z) \ , \tag{12.90}$$

giving a surface energy (per unit mass)$\sim h_m\mathcal{E}/\rho$. Making use of the representation $\delta^{'2}(z) \simeq 1/h_m^4$, this surface energy is proportional to h^2/h_m^3, while the corresponding energy of the incident wave goes like h_m/λ^2; the ratio of the two quantities is of the order of $h^2\lambda^2/h_m^4$. We can see that that this ratio may acquire large values, even for

$h \ll h_m$ (perturbation criterion satisfied), for $\lambda \gg h_m$. Therefore, the surface waves may store an appreciable amount of energy, as a result of their localization. This phenomenon is related to the discontinuites experienced by the strain tensor along the direction perpendicular to the surface.

12.2.6 Particular cases

From equations (12.79), (12.82) and (12.86) we can get the reflection coefficients, related to the energy, of the waves scattered back in the body. Their general characteristic is the directionality effects. The derivation of these coefficients is complicated in the general case, where we should fix the amplitudes of the original $\mathbf{u}_0$-waves according to the nature of these waves and the boundary conditions. Another complication arises from the fact that we should "renormalize" the amplitudes of the reflected original $\mathbf{u}_0$-waves such as to include (accomodate) the scattered waves in the boundary conditions (a procedure specific to theoretical-perturbation calculations). We limit ourselves here to give the reflection coefficients for a few particular cases.

First, one of the simplest case is an original s-wave, described by

$$\mathbf{u}_0 = 2(0,\, u_0,\, 0)\cos\kappa_0 z \cdot e^{-i\omega t + i\mathbf{k}_0\mathbf{r}} \;, \tag{12.91}$$

where $\mathbf{k}_0$ is directed along the x-axis. Making use of equation (12.89), the energy density (per unit mass) of the incident wave in equation (12.91) is $\mathcal{E}_0/\rho = \omega^2 u_0^2$. We must compute the projections $v_{01,2}$ of the amplitude of this wave on $\mathbf{k} = \mathbf{k}_0 + \mathbf{q}$ and $\mathbf{k}_\perp$. Introducing the angle α between $\mathbf{q}$ and $\mathbf{k}_0$, we get $v_{01} = 2u_0 q\sin\alpha/k$ and $v_{02} = 2u_0(k_0 + q\cos\alpha)/k$ (and, of course, $w_0 = 0$). We can see, from equations (12.79), (12.82) and (12.86), that an incident s-wave produce both s- and p- scattered transverse waves as well as a scattered longidudinal wave, due to the surface inhomogeneities. Making use of these equations we compute easily the amplitudes of these waves and get the reflection coefficients

$$\begin{gathered} R_s = \eta_t \tfrac{h\omega^2}{4v_t^2\kappa k}(k_0 + q\cos\alpha) \;, \\ R_p = \eta_t \tfrac{h\omega q}{4v_t k}\sin\alpha \;,\; R_l = \eta_l \tfrac{h\omega q}{4v_l k}\sin\alpha \;. \end{gathered} \tag{12.92}$$

The energy density carried on by these waves is given by $\mathcal{E}_{s,p,l}/\mathcal{E}_0 = R^2_{s,p,l}$. We stress upon the complicated direction-dependence (angle α) of these reflection coefficients, included both in κ and k. The formulae given by equations (12.92) become more simple for normal incidence ($\mathbf{k}_0 = 0$).

For normal incidence there is another simple case concerning longitudinal waves described by

$$\mathbf{u}_0 = 2(0,\, 0,\, u_0) \cos \kappa_0^{'} z \cdot e^{-i\omega t} \;\; , \tag{12.93}$$

where $\kappa_0^{'} = \omega/v_l$. The energy density per unit mass of this incident wave is $E_0/\rho = \omega^2 u_0^2$. According to equations (12.79), (12.82) and (12.86), the scattered waves in this case are a p-wave and a longitudinal wave. Their reflection coefficients are much more simple now,

$$R_p = \eta_t \frac{h\omega q}{4v_t\kappa} \;,\;\; R_l = \eta_l \frac{h\omega\kappa^{'}}{4v_l q} \;\; . \tag{12.94}$$

The squares of these coefficients give the fraction of energy carried on by these waves.

It is worth stressing that all the above formulae are valid only for κ, k, $q \neq 0$ (non-vanishing denominators).

We can see from the above particular cases, as well as from the general equations (12.79), (12.82) and (12.86), that the total amount of energy carried on diffusively by the waves scattered by the surface roughness implies sums of the form $\sum_{\mathbf{q}} |h(\mathbf{q})|^2 f(\mathbf{q})$, where $h(\mathbf{q})$ is the Fourier transform of the roughness function $h(\mathbf{r})$ and $f(\mathbf{q})$ are specific functions corresponding to the waves' nature (factors implying k, κ, $\kappa^{'}$, etc). Qualitatively, in order to maximize this energy, it is necessary, apart from particular cases of gratings (one, or a few wavevectors $\mathbf{q}$), to include as many Fourier components as possible, *i.e.* the surface should be as rough as possible in order to have a good attenuation, a reasonably expected result.

In conclusion, we may say that we have introduced a model of inhomogeneous surface distribution of defects for a semi-infinite isotropic elastic body and solved the wave equations for the elastic waves scattered by this surface roughness in the first-order approximation with respect to the magnitude of the defects distribution. The scattered waves are of two kinds: waves localized (and propagating only) on the surface, given by equations (12.80) and (12.84), and scattered waves reflected back in the body by the surface inhomogeneities, both transverse, as given by equations (12.79) and (12.82), and longitudinal, as given by equation (12.86). The latter may become confined to the surface (damped surface waves) for an enhanced roughness (large wavevectors q). The reflected waves are absent for a homogeneous roughness ($\eta_{t,l} = 0$), where only the localized waves survive.

12.3 Bulk inhomogeneities

12.3.1 Introduction

A new method is introduced herein for estimating the effects of the inhomogeneities on the propagation of the elastic waves in isotropic bodies. The method is based on the Kirchhoff potentials. It is aplied here for estimating the effect of a static density inhomogeneity, either extended or localized, on the elastic waves propagating in an infinite, or a semi-infinite (half-space) body. For a semi-infinite body the method leads to coupled integral equations, which are solved. It is shown that such a density inhomogeneity may renormalize the waves velocity, or may even produce dispersive waves, depending on the geometry of the body and the spatial extension of the inhomogeneity.[19] The method can be extended to other types of geometries or inhomogeneities, as, for instance, those occurring in the elastic constants. The same method is employed for the propagation of sound in fluids.[20]

The effect of the inhomogeneities on the propagation of the elastic waves in structures with special, restricted geometries has always enjoyed a great deal of interest. Apart from its practical importance in engineering, the problem is particularly relevant for the effect the seismic waves may have on the Earth's surface.[21] The propagation

[19] B. F. Apostol, "The effect of the inhomogeneities on the propagation of elastic waves in isotropic bodies", Mech. Res. Commun. **37** 458 (2010).

[20] B. F. Apostol, "Scattering of longitudinal waves (sound) by defects in fluids. Rough surface", Centr. Eur. J. Phys. **11** 1036 (2013).

[21] K. E. Bullen, *An Introduction to the Theory of Seismology*, Cambridge University Press, Cambridge (1976); K. Aki and P. G. Richards, *Quantitative*

of elastic waves in bodies with finite, special geometries, like, for instance, a semi-infinite space, poses certain technical problems. We present herein a new method of dealing with elastic waves in isotropic media, borrowed from electromagnetism. The method is based on Kirchhoff retarded potentials for the wave equation. We analyze the change produced in the eigenfrequencies of the elastic modes by static density inhomogeneities of a certain spatial extent, distributed in an infinite, or a semi-infinite (half-space) isotropic body.

As it is well known, the propagation of the elastic waves in an isotropic body is governed by the equation of motion

$$\rho\ddot{\mathbf{u}} = \mu\Delta\mathbf{u} + (\lambda+\mu)\, grad\, div\mathbf{u} \ , \tag{12.95}$$

where ρ is the body density, $\mathbf{u}$ is the field displacement and $\lambda\,,\mu$ are the Lame coefficients. We leave aside the external forces and write this equation in the form

$$\frac{1}{v_t^2}\ddot{\mathbf{u}} - \Delta\mathbf{u} = q\cdot grad\, div\mathbf{u} \ , \tag{12.96}$$

where $v_t = \sqrt{\mu/\rho}$ is the velocity of the transverse waves, $q = v_l^2/v_t^2 - 1$ and $v_l = \sqrt{(\lambda+2\mu)/\rho}$ is the velocity of the longitudinal waves. As it is well-known, for reasons of stability, the inequality $q > 1/3$ (actually $q > 1$ for real bodies) holds. A particular solution of equation (12.96) is given by the well-known Kirchhoff potential[22]

$$\mathbf{u}(\mathbf{R},t) = \tfrac{q}{4\pi}\int d\mathbf{R}'\tfrac{grad\, div\mathbf{u}(\mathbf{R}',t-\left|\mathbf{R}-\mathbf{R}'\right|/v_t)}{|\mathbf{R}-\mathbf{R}'|} \ . \tag{12.97}$$

Indeed, making use of Fourier transforms and using also the well-known integral

$$\int d\mathbf{R}\frac{e^{i\mathbf{K}\mathbf{R}+i\omega R/v_t}}{R} = -\frac{4\pi v_t^2}{\omega^2 - v_t^2K^2} \ , \tag{12.98}$$

we get the eigenvalue equation

$$\left(-\rho\omega^2 + \mu K^2\right)\overline{\mathbf{u}} = -(\lambda+\mu)\,(\mathbf{K}\overline{\mathbf{u}})\mathbf{K} \ , \tag{12.99}$$

Seismology, Theory and Methods, Freeman, San Francisco (1980); A. Ben-Menahem and J. D. Singh, *Seismic Waves and Sources*, Springer, New York (1981).

[22] M. Born and E. Wolf, *Principles of Optics,* Pergamon, London (1959).

where ω denotes the frequency, $\mathbf{K}$ is the wavevector and $\overline{\mathbf{u}}(\mathbf{K},\omega)$ is the Fourier transform of $\mathbf{u}(\mathbf{R},t)$. One can check immediately that equation (12.99) gives the well-known transverse and longitudinal elastic waves propagating in an infinite isotropic body.

For a semi-infinite body extending over the region $z>0$, with a free surface in the (x,y)-plane $z=0$, we use

$$\mathbf{u}\rightarrow\mathbf{u}\theta(z)=(\mathbf{v},u_3)\theta(z) \tag{12.100}$$

for the displacement field, where $\theta(z)=1$ for $z>0$, $\theta(z)=0$ for $z<0$ is the step function, $\mathbf{v}$ is the (x,y) in-plane component and u_3 is the normal-to-surface component of the displacement (directed along the z-coordinate). We note that we employ in fact distributions (in the sense of generalized functions) like $\theta(z)$ or $\delta(z)$, etc, instead of usual functions. For the function $\mathbf{u}$ in $\mathbf{u}\theta(z)$ (defined over the entire space) we use Fourier transforms of the type

$$\mathbf{u}(\mathbf{r},z;t)=\sum_{\mathbf{k}}\int d\omega\widetilde{\mathbf{u}}(\mathbf{k}\omega;z)e^{i\mathbf{kr}}e^{-i\omega t}\ \ , \tag{12.101}$$

where $\mathbf{R}=(\mathbf{r},z)$ and $\widetilde{\mathbf{u}}(\mathbf{k}\omega;z)$ is the (partial) Fourier transform of $\mathbf{u}(\mathbf{r},z;t)$ with respect to $\mathbf{r}$ and t. The divergence occurring in equation (12.97) can then be written as

$$div\mathbf{u}=\left(div\mathbf{v}+\frac{\partial u_3}{\partial z}\right)\theta(z)+u_3(0)\delta(z)\ \ , \tag{12.102}$$

where we can see the occurrence of the surface term $u_3(0)=u_3(z=0)$. The gradient can be computed similarly, by using the Fourier transform given by equation (12.101).

12.3.2 Inhomogeneities

We assume a certain region in the body, whose shape and extension is described by a function $g(\mathbf{r},z)$, where the density of the body is modified according to

$$\rho\rightarrow\rho+\rho g(\mathbf{r},z)\ . \tag{12.103}$$

We employ equation (12.103) for describing an inhomogeneity in the body. It is easy to see that this change in density introduces a new source term in equation (12.96), which can be written as

$$-\frac{1}{v_t^2}g(\mathbf{r},z)\ddot{\mathbf{u}}(\mathbf{r},z;t)=\sum_{\mathbf{k}}\int d\omega\frac{\omega^2}{v_t^2}\widetilde{\mathbf{h}}(\mathbf{k}\omega;z)e^{i\mathbf{kr}}e^{-i\omega t}\ , \qquad (12.104)$$

where

$$\widetilde{\mathbf{h}}(\mathbf{k}\omega;z)=\sum_{\mathbf{k}_1}\widetilde{g}(\mathbf{k}-\mathbf{k}_1,z)\widetilde{\mathbf{u}}(\mathbf{k}_1\omega;z)\ . \qquad (12.105)$$

Consequently, equation (12.97) becomes

$$\begin{aligned}\mathbf{u}(\mathbf{R},t)=&\tfrac{q}{4\pi}\int d\mathbf{R}'\frac{grad\,div\mathbf{u}(\mathbf{R}',t-\left|\mathbf{R}-\mathbf{R}'\right|/v_t)}{|\mathbf{R}-\mathbf{R}'|}-\\ &-\tfrac{1}{4\pi v_t^2}\int d\mathbf{R}'\frac{g(\mathbf{r}',z')\ddot{\mathbf{u}}(\mathbf{R}',t-\left|\mathbf{R}-\mathbf{R}'\right|/v_t)}{|\mathbf{R}-\mathbf{R}'|}\ .\end{aligned} \qquad (12.106)$$

Making use of the representations given above, and after performing conveniently a few integrations by parts, equation (12.106) can be simplified appreciably. The intervening integrals can be performed straightforwardly. They reduce to the known integral[23]

$$\int_{|z|}^{\infty}dxJ_0\left(k\sqrt{x^2-z^2}\right)e^{i\omega x/v_t}=\frac{i}{\kappa_0}e^{i\kappa_0|z|}\ , \qquad (12.107)$$

where J_0 is the Bessel function of the first kind and zeroth order and

$$\kappa_0=\sqrt{\frac{\omega^2}{v_t^2}-k^2}\ . \qquad (12.108)$$

We get the system of coupled integral equations

$$\begin{aligned}\widetilde{\mathbf{v}}(\mathbf{k}\omega;z)=&-\tfrac{iq\mathbf{k}}{2\kappa_0}\int_0 dz'\mathbf{k}\widetilde{\mathbf{v}}(\mathbf{k}\omega;z')e^{i\kappa_0\left|z-z'\right|}-\\ &\tfrac{q\mathbf{k}}{2\kappa_0}\tfrac{\partial}{\partial z}\int_0 dz'\widetilde{u}_3(\mathbf{k}\omega;z')e^{i\kappa_0\left|z-z'\right|}+\\ &+\tfrac{i\omega^2}{2v_t^2\kappa_0}\int_0 dz'\widetilde{\mathbf{h}}_{\|}(\mathbf{k}\omega;z')e^{i\kappa_0\left|z-z'\right|}\end{aligned} \qquad (12.109)$$

[23] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, NY (2000), pp. 714-715, 6.677; 1,2.

and

$$\widetilde{u}_3(\mathbf{k}\omega;z) = -\frac{q}{2\kappa_0}\frac{\partial}{\partial z}\int_0 dz'\mathbf{k}\widetilde{\mathbf{v}}(\mathbf{k}\omega;z')e^{i\kappa|z-z'|}+$$

$$+\frac{iq}{2\kappa_0}\frac{\partial^2}{\partial z^2}\int_0 dz'\widetilde{u}_3(\mathbf{k}\omega;z')e^{i\kappa|z-z'|}+ \qquad (12.110)$$

$$+\frac{i\omega^2}{2v_t^2\kappa_0}\int_0 dz'\widetilde{h}_3(\mathbf{k}\omega;z')e^{i\kappa_0|z-z'|} \ ,$$

where $\widetilde{\mathbf{h}}_{||}$ is the in-plane component of the vector $\widetilde{\mathbf{h}}$ defined by equation (12.105) and $\widetilde{h}_3$ is its component along the z-direction. The details for deriving these equations are given in **Appendix**.

It is convenient to introduce the notations $\widetilde{v}_1 = \mathbf{k}\widetilde{\mathbf{v}}/k$, $\widetilde{v}_2 = \mathbf{k}_\perp\widetilde{\mathbf{v}}/k$, and similar ones for the vector $\widetilde{\mathbf{h}}$, where $\mathbf{k}_\perp$ is a vector perpendicular to $\mathbf{k}$, $\mathbf{k}\mathbf{k}_\perp = 0$, and of the same magnitude k. Under these conditions equation (12.109) for $\widetilde{v}_2$ reduces to

$$\widetilde{v}_2(\mathbf{k}\omega;z) = \frac{i\omega^2}{2v_t^2\kappa_0}\int_0 dz'\widetilde{h}_2(\mathbf{k}\omega;z')e^{i\kappa_0|z-z'|} \ . \qquad (12.111)$$

This equation corresponds to the transverse wave polarized perpendicular to the plane of propagation (it is known in electromagnetism as the s-wave, from the German "senkrecht" which means "perpendicular"). Taking the second derivative with respect to z in this equation we get

$$\frac{\partial^2\widetilde{v}_2}{\partial z^2} = -\kappa_0^2\widetilde{v}_2 - \frac{\omega^2}{v_t^2}\widetilde{h}_2 \ . \qquad (12.112)$$

Here, it is worth noting the non-invertibility of the (second) derivative and the integral in equation (12.111), as a result of the discontinuity in the derivative of the function $e^{i\kappa_0|z-z'|}$ for $z = z'$. In equation (12.112) we perform a Fourier transform with respect to the coordinate z. Introducing the wavevectors $\mathbf{K} = (\mathbf{k},\kappa)$ and $\mathbf{K}_1 = (\mathbf{k},\kappa_1)$ and making use of equation (12.108), equation (12.112) becomes

$$\left(\frac{\omega^2}{v_t^2} - K^2\right)\overline{v}_2(\mathbf{K}\omega) = -\frac{\omega^2}{v_t^2}\sum_{\mathbf{K}_1}\overline{g}(\mathbf{K}-\mathbf{K}_1)\overline{v}_2(\mathbf{K}_1\omega) \ . \qquad (12.113)$$

We assume first that function $g(\mathbf{R})$ is a constant, $g(\mathbf{R}) = g$. Then, $\overline{g}(\mathbf{K}) = g\delta_{\mathbf{K},0}$ and equation (12.113) gives the frequency

$$\omega = \frac{v_t}{\sqrt{1+g}} K \ , \qquad (12.114)$$

an expected result, which shows that the wave velocity is renormalized as a consequence of the change in density, as described by the parameter g. Second, we assume that the function $g(\mathbf{R})$ is localized at some position $\mathbf{R}_0$ in the body over a small spatial range of linear extension a. Then, its Fourier transform can be taken almost constant, $\overline{g}(\mathbf{K}) \simeq ga^3/V$, over a range $\sim 1/a$, where V is the volume of the Fourier integration and $g = g(\mathbf{R}_0)$. Under these conditions we get from equation (12.113) the dispersion relation

$$1 = -\frac{\omega^2 g a^3}{v_t^2 V} \sum_{\mathbf{K}} \frac{1}{\omega^2/v_t^2 - K^2} \ . \qquad (12.115)$$

For small values of g the solutions of this equation are given by

$$\omega^2/v_t^2 = K^2 - \frac{g\omega^2}{6\pi^2 v_t^2} = K^2 - \frac{g}{6\pi^2}K^2 + ..., \qquad (12.116)$$

whence, in the first approximation, we get another renormalizaton of the wave velocity

$$v_t \to v_t \left(1 - \frac{g}{12\pi^2}\right) \ . \qquad (12.117)$$

More specifically, we may take for the localized function $g(\mathbf{R})$ a Gaussian normal distribution of the form $g(\mathbf{R}) = g\exp\left(-\left|\mathbf{R}-\mathbf{R}_0\right|/2a^2\right)$, centered at $\mathbf{R}_0$ and of standard deviation a. As it is well known, its Fourier transform is $\overline{g}(\mathbf{K}) = (g/V)(2\pi a^2)^{3/2}\exp\left(-i\mathbf{K}\mathbf{R}_0\right)\cdot$, $\cdot\exp\left(-K^2a^2/2\right)$ which is, essentially, another Gaussian distribution, centered at $\mathbf{K} = 0$ and of standard deviation $1/a$. In this case, the correction term $\left(g/6\pi^2\right)K^2$ in equation (12.116) acquires an additional factor $(2\pi)^{3/2}$. However, we emphasize here that all these numerical results are only qualitative estimations.

We note that the renormalization given by equation (12.117) does not depend on the spatial extension of the function $g(\mathbf{R})$. We also note that these results are the same for an infinite body. For a general function $g(\mathbf{R})$ we may obtain a renormalization of the velocity comprised between the two limiting cases given above by equations (12.114) and (12.117). As one can see, there is a qualitative resemblance between these two results (for instance, equation (12.114) can also be written as $v_t \to v_t(1-g/2)$). But we must keep in mind that all these are only approximate, qualitative estimations. The exact solution would imply solving the integral equation (12.113) (a homogeneous Fredholm equation of the second kind), which, for a general kernel $\overline{g}(\mathbf{K}-\mathbf{K}_1)$, is a difficult problem. Generally speaking, it implies finding out the eigenfunctions and eigenvalues of the kernel. Under certain conditions, we may try an iterative technique, which may offer an insight into the qualitative behaviour of the solution: the dispersion relation $\omega(\mathbf{K})$ will exhibit both dispersion and anisotropy, and the waves will get anisotropic, dispersive group velocities. They may, more appropriately, be then viewed as dispersive, anisotropic wave packets.

It is also interesting the case where the localized inhomogeneities are randomly distributed in the whole body, *i.e.* the function $g(\mathbf{R})$ is given by

$$g(\mathbf{R}) = \sum_{i=1}^{N} g_i(\mathbf{R}-\mathbf{R}_{0i}) \ , \tag{12.118}$$

where $g_i(\mathbf{R}-\mathbf{R}_{0i})$ is a function of strength g_i localized over the volume a_i^3 centered at $\mathbf{R}_{0i}$ and N denotes the number of these inhomogeneities. The Fourier transform is given then approximately by $g_i(\mathbf{K}) \simeq g_i a_i^3/V$, which extends over a volume $\sim 1/a_i^3$. By repeating the above calculations for equation (12.113) we get a renormalization of the velocity given by

$$v_t \to v_t \left(1 - \frac{1}{12\pi^2}\sum_i g_i\right) = v_t\left(1 - \frac{N\overline{g}}{12\pi^2}\right) \ , \tag{12.119}$$

where $\overline{g}$ is the mean strength, as expected. If the inhomogeneities are distributed in a regular, periodic array, then the problem becomes more complicated, because the Fourier transforms will then be peaked at all the reciprocal vectors of the array. The integral equation (12.113) is then replaced by another integral equation, implying summation over all the reciprocal vectors, but with similar (common) kernels in all the terms of the summation. The qualitative behaviour of the solutions of such equations are known from the theory of energy bands in solids (or the propagation of light in periodic media[24]): due to the multiple reflections, the waves may form stationary waves and the frequencies ω will be distributed in bands, separated by frequency gaps. However, such subjects will take the present discussion too far.

We can also consider a layer of thickness a, *i.e.* take $g(\mathbf{R}) = g(z-z_0)$, where $g(z-z_0)$ is a function localized over the thickness a around z_0. Its Fourier transform is $\overline{g}(\mathbf{k},\kappa) \simeq (ga/L)\,\delta_{\mathbf{k},0}$, where L is the length of the Fourier integration along the z-direction and $\overline{g}(\mathbf{k},\kappa)$ extends over a range $\sim 1/a$ as a function of κ. We note that function $g(\mathbf{R}) = g(z-z_0)$ does not depend on $\mathbf{r}$. Of course, the definition of such a (full) Fourier transform is

$$\begin{gathered}\mathbf{u}(\mathbf{r},z;t) = \mathbf{u}(\mathbf{R},t) = \sum_{\mathbf{k}\kappa} \int d\omega \overline{\mathbf{u}}(\mathbf{k},\kappa,\omega) e^{i\mathbf{kr}} e^{i\kappa z} = \\ = \sum_{\mathbf{K}} \int d\omega \overline{\mathbf{u}}(\mathbf{K},\omega) e^{i\mathbf{KR}} \ ,\end{gathered} \tag{12.120}$$

(compare with equation (12.101)), where the summations (integrations) over $\mathbf{k}$, κ and ω extend over the entire space. The velocity is then renormalized according to

$$v_t \rightarrow v_t \left(1 - \frac{g}{4\pi}\right) \ . \tag{12.121}$$

We turn now to equation (12.109) written for $\widetilde{v}_1$ and equation (12.110) for $\widetilde{u}_3$. We leave aside arguments $\mathbf{k}\,,\omega$ for simplicity, and preserve explicitly only the argument z. It is easy to see that these two equations imply

$$\widetilde{u}_3(z) = -\frac{i}{k}\frac{\partial \widetilde{v}_1}{\partial z} - \frac{\omega^2}{2v_t^2\kappa_0 k}\frac{\partial \widetilde{H}_1}{\partial z} + \frac{i\omega^2}{2v_t^2\kappa_0}\widetilde{H}_3(z) \ , \tag{12.122}$$

where

$$\widetilde{H}_{1,3}(z) = \int_0 dz' \widetilde{h}_{1,3}(z') e^{i\kappa_0 |z-z'|} \ . \tag{12.123}$$

[24] L. Brillouin and M. Parodi, *Propagation des Ondes dans les Milieux Periodiques*, Dunod, Paris (1956).

We introduce $\widetilde{u}_3(z)$ as given by equation (12.122) in equation (12.109) for $\widetilde{v}_1(z)$ and take the second derivative in the resulting equation. We get

$$\frac{\partial^2 \widetilde{v}_1}{\partial z^2} + \kappa_0^{'2} \widetilde{v}_1 = \frac{i\omega^2}{2v_t^2 \kappa_0} \left(\frac{\partial^2 \widetilde{H}_1}{\partial z^2} + \frac{\kappa_0^2 v_t^2}{v_l^2} \widetilde{H}_1 \right) + \frac{qk\omega^2}{2v_l^2 \kappa_0} \frac{\partial \widetilde{H}_3}{\partial z} \ , \quad (12.124)$$

where

$$\kappa_0^{'} = \sqrt{\frac{\omega^2}{v_l^2} - k^2} \ . \quad (12.125)$$

We introduce Fourier transforms with respect to the z-coordinate both in equation (12.122) and equation (12.124). The Fourier transforms of the functions $\widetilde{H}_{1,3}(z)$ are

$$\overline{H}_{1,3}(\kappa) = -\frac{2i\kappa_0}{\kappa^2 - \kappa_0^2} \overline{h}_{1,3}(\kappa) \quad (12.126)$$

for $\kappa \neq \kappa_0$. Restoring the arguments, $\overline{h}_1(\kappa)$ is written, by equation (12.105), as

$$\overline{h}_1(\mathbf{K}) = \sum_{\mathbf{K}_1} \overline{g}(\mathbf{K} - \mathbf{K}_1) \overline{v}_1(\mathbf{K}_1) \ ; \quad (12.127)$$

a similar expression holds for $\overline{h}_3$. Doing so, we get two coupled equations

$$\begin{aligned} &\overline{u}_3(\mathbf{K}) - \tfrac{\kappa}{k} \overline{v}_1(\mathbf{K}) + \tfrac{\omega^2}{\omega^2 - v_t^2 K^2} \cdot \\ &\cdot \textstyle\sum_{\mathbf{K}_1} \overline{g}(\mathbf{K} - \mathbf{K}_1) \left[\overline{u}_3(\mathbf{K}_1) - \tfrac{\kappa}{k} \overline{v}_1(\mathbf{K}_1) \right] = 0 \end{aligned} \quad (12.128)$$

and

$$\begin{aligned} &(\omega^2 - v_t^2 K^2)(\omega^2 - v_l^2 K^2) \overline{v}_1(\mathbf{K}) + \\ &+ \omega^2 (\omega^2 - v_l^2 \kappa^2 - v_t^2 k^2) \textstyle\sum_{\mathbf{K}_1} \overline{g}(\mathbf{K} - \mathbf{K}_1) \overline{v}_1(\mathbf{K}_1) + \\ &+ q v_t^2 \kappa k \omega^2 \textstyle\sum_{\mathbf{K}_1} \overline{g}(\mathbf{K} - \mathbf{K}_1) \overline{u}_3(\mathbf{K}_1) = 0 \ . \end{aligned} \quad (12.129)$$

In analyzing these equations we proceed as before. For a constant function $g(\mathbf{R}) = g$, whose Fourier transform is $\overline{g}(\mathbf{K}) = g\delta_{\mathbf{K},0}$, equations (12.128) and (12.129) give two types of waves. For the longitudinal wave, $\overline{u}_3 = \kappa \overline{v}_1 / k$, equation (12.128) is satisfied identically, while from equation (12.129) we get a renormalization of the velocity v_l which is the same as that given above by equation (12.114). For the transverse wave $\overline{u}_3 = -k\overline{v}_1/\kappa$ (p-wave, whose polarization lies in the plane of propagation) we get from equations (12.128) and (12.129) the same renormalization of the velocity v_t as that given by equation (12.114).

We assume now a function $g(\mathbf{R})$ localized at some position $\mathbf{R}_0$ within the body and extending over a range $\sim a$. Its Fourier transform can be taken as $\overline{g}(\mathbf{K}) \simeq ga^3/V$ for $\mathbf{K}$ extending over a range $\sim 1/a$ and $g = g(\mathbf{R}_0)$. It is easy to see that, according to equations (12.128) and (12.129), the velocity v_t is not renormalized in the first order of the (small) parameter g, but the velocity v_l acquires a renormalization given by

$$v_l \to v_l \left(1 - \frac{g}{36\pi^2}\right) \ . \tag{12.130}$$

Similarly, for a layer of thickness a the velocity v_t is not renormalized in the first order of the parameter g but the frequency of the longitudinal waves becomes

$$\omega = v_l K \left(1 - \frac{gak}{4}\right) \ ; \tag{12.131}$$

we can see that the longitudinal waves become dispersive in this case.

For comparison we give here the results for a density inhomogeneity in an infinite elastic body. By using Fourier transforms, equation (12.106) leads to

$$\overline{\mathbf{u}}(\mathbf{K}\omega) = \frac{qv_t^2}{\omega^2 - v_t^2 K^2}(\mathbf{K}\overline{\mathbf{u}})\mathbf{K} - \frac{\omega^2}{\omega^2 - v_t^2 K^2}\overline{\mathbf{h}}(\mathbf{K}\omega) \ , \tag{12.132}$$

where

$$\overline{\mathbf{h}}(\mathbf{K}\omega) = \sum_{\mathbf{K}_1} \overline{g}(\mathbf{K} - \mathbf{K}_1)\overline{\mathbf{u}}(\mathbf{K}_1\omega) \tag{12.133}$$

and we have used the integral given by equation (12.98). Equation (12.132) reduces to

$$\overline{u}_{1,2}(\mathbf{K}\omega) + \frac{\omega^2}{\omega^2 - v_{l,t}^2 K^2} \sum_{\mathbf{K}_1} \overline{g}(\mathbf{K} - \mathbf{K}_1)\overline{u}_{1,2}(\mathbf{K}_1\omega) = 0 \tag{12.134}$$

for the longitudinal waves $\overline{u}_1 = \overline{\mathbf{u}}\mathbf{K}/K$ (velocity v_l) and transerse waves $\overline{u}_2 = \overline{\mathbf{u}}\mathbf{K}_\perp/K$ (velocity v_t), where $\mathbf{K}_\perp$ is a vector perpendicular to the wavevector $\mathbf{K}$, $\mathbf{K}\mathbf{K}_\perp = 0$, and of the same magnitude K. Both equations (12.134) lead to a dispersion equation of the same form as the one corresponding to the s-wave (equation (12.113)). For an extended inhomogeneity both $v_{t,l}$ are renormalized according to equation (12.114), for a localized inhomogeneity both velocities are renormalized according to equation (12.117). This is different than the semi-infinite body (compare with equation (12.130)).

In conclusion we may say that we have introduced herein a new method, based on the Kirchhoff electromagnetic potentials, to estimate the effects of density inhomogeneities on the propagation of the elastic waves in isotropic bodies. We have applied this method both to an infinite body and a semi-infinite (half-space) body. For an infinite body a density inhomogeneity renormalizes the velocity of the transverse and longitudinal waves. We have estimated this effect both for an extended and a localized inhomogeneity, or for a layer, assuming that the strength of the inhomogeneity is small (parameter g). For a semi-infinite body the present method leads to coupled integral equations which we have solved. The transverse s-wave is affected in the same manner as in an infinite body, and this holds also for all the waves for an extended inhomogeneity, as expected. For a localized inhomogeneity the transverse p-wave is affected in the second-order of the parameter g, while the longitudinal wave undergoes a renormalization of velocity (different than in an infinite body). In addition, for a layer inhomogeneity, the longitudinal waves become dispersive.

12.3.3 Appendix: equations (12.109) and (12.110)

Let us denote by

$$\mathbf{F}(\mathbf{r}, z; t) = -\frac{1}{4\pi v_t^2}\int d\mathbf{R}'\frac{g(\mathbf{r}', z')\ddot{\mathbf{u}}(\mathbf{R}', t - |\mathbf{R} - \mathbf{R}'|/v_t)}{|\mathbf{R} - \mathbf{R}'|} \quad (12.135)$$

the second term in the *rhs* of equation (12.106), where $\mathbf{R} = (\mathbf{r}, z)$ and $\mathbf{R}' = (\mathbf{r}', z')$. We note that the integration here extends over the whole space (according to the definition of the Kirchhoff potentials). First, we replace $\mathbf{u}$ by $\mathbf{u}\theta(z)$, which will restrict the integration with respect to z' to $0 < z' < \infty$. Second, we perform the Fourier transform with respect to the time, according to equation (12.101), which will bring a factor $-\omega^2$. Then, we introduce the spatial Fourier transforms (according to the same equation (12.101)) and get

$$\widetilde{\mathbf{F}}(\mathbf{r}, z; \omega) = \frac{\omega^2}{4\pi v_t^2} \sum_{\mathbf{k}_1\mathbf{k}_2} \int_0^\infty dz' \int d\mathbf{r}' \frac{\widetilde{g}(\mathbf{k}_2, z')\widetilde{\mathbf{u}}(\mathbf{k}_1\omega; z')}{\sqrt{(\mathbf{r}-\mathbf{r}')^2+(z-z')^2}} \times$$
$$\times e^{i\frac{\omega}{v_t}\sqrt{(\mathbf{r}-\mathbf{r}')^2+(z-z')^2}} e^{i(\mathbf{k}_1+\mathbf{k}_2)\mathbf{r}'} \; . \qquad (12.136)$$

In this equation we introduce the new variable $\mathbf{r}_1 = \mathbf{r}' - \mathbf{r}$ and put $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}$. We get immediately the Fourier transform

$$\widetilde{\mathbf{F}}(\mathbf{k}\omega; z) = \frac{\omega^2}{4\pi v_t^2} \sum_{\mathbf{k}_1} \int_0^\infty dz' \int d\mathbf{r}_1 \frac{\widetilde{g}(\mathbf{k}-\mathbf{k}_1, z')\widetilde{\mathbf{u}}(\mathbf{k}_1\omega; z')}{\sqrt{r_1^2+(z-z')^2}} \times$$
$$\times e^{i\frac{\omega}{v_t}\sqrt{r_1^2+(z-z')^2}} e^{i\mathbf{k}\mathbf{r}_1} \; . \qquad (12.137)$$

Now, by successive integrations, we have

$$\int d\mathbf{r}_1 \frac{e^{i\frac{\omega}{v_t}\sqrt{r_1^2+z^2}}}{\sqrt{r_1^2+z^2}} e^{i\mathbf{k}\mathbf{r}_1} =$$
$$= 2\pi \int_0^\infty dr_1 r_1 \frac{e^{i\frac{\omega}{v_t}\sqrt{r_1^2+z^2}}}{\sqrt{r_1^2+z^2}} J_0(kr_1) = \qquad (12.138)$$
$$= 2\pi \int_{|z|}^\infty dx J_0(k\sqrt{x^2-z^2}) e^{i\frac{\omega}{v_t}x} = \frac{2\pi i}{\kappa_0} e^{i\kappa_0|z|} \; ,$$

according to equation (12.107), where $\kappa_0 = \sqrt{\omega^2/v_t^2 - k^2}$ (equation (12.108)). In equation (12.138) J_0 is the Bessel function of the first kind and zeroth order and we made the change of variable $r_1^2+z^2 = x^2$. This result will be used in all the subsequent calculations. We may replace now the integral with respect to $\mathbf{r}_1$ in equation (12.137) by this result and get

$$\widetilde{\mathbf{F}}(\mathbf{k}\omega; z) = \frac{i\omega^2}{2v_t^2\kappa_0} \cdot$$
$$\cdot \sum_{\mathbf{k}_1} \int_0^\infty dz' \widetilde{g}(\mathbf{k}-\mathbf{k}_1, z')\widetilde{\mathbf{u}}(\mathbf{k}_1\omega; z') e^{i\kappa_0|z-z'|} \; , \qquad (12.139)$$

or

$$\widetilde{\mathbf{F}}(\mathbf{k}\omega;z)=\frac{i\omega^2}{2v_t^2\kappa_0}\int_0^\infty dz'\widetilde{\mathbf{h}}(\mathbf{k}\omega;z')e^{i\kappa_0|z-z'|} \ , \tag{12.140}$$

according to definition (12.105). We can already recognize the last term in the *rhs* of equations (12.109) and (12.110).

Now, we pass to the first term in the *rhs* of equation (12.106). First, we replace here $\mathbf{u}$ by $\mathbf{u}\theta(z)$. Second, we note that this term is computed for $\mathbf{u}(\mathbf{R}',t')$, where the time t' is then replaced by the retarded time $t-|\mathbf{R}-\mathbf{R}'|/v_t$ (according to the definition of the retarded Kirchhoff potentials). Making use of equations (12.100) and (12.102), and introducing the Fourier transform with respect to $\mathbf{r}$, we get

$$div\mathbf{u}=\sum_{\mathbf{k}}\left[\left(i\mathbf{k}\mathbf{v}+\frac{\partial u_3}{\partial z}\right)\theta(z)+u_3(0)\delta(z)\right]e^{i\mathbf{k}\mathbf{r}} \tag{12.141}$$

and

$$\begin{gathered}(grad\cdot div\mathbf{u})_{\parallel}=\\ =\sum\nolimits_{\mathbf{k}}\left[i\mathbf{k}\left(i\mathbf{k}\mathbf{v}+\frac{\partial u_3}{\partial z}\right)\theta(z)+i\mathbf{k}u_3(0)\delta(z)\right]e^{i\mathbf{k}\mathbf{r}}\end{gathered} \tag{12.142}$$

for the in-plane component of the gradient and

$$\begin{gathered}(grad\cdot div\mathbf{u})_3=\\ =\sum\nolimits_{\mathbf{k}}\left[\left(ik\frac{\partial v}{\partial z}+\frac{\partial^2 u_3}{\partial z^2}\right)\theta(z)+\left(i\mathbf{k}\mathbf{v}+\frac{\partial u_3}{\partial z}\right)\delta(z)\right]e^{i\mathbf{k}\mathbf{r}}+\\ +\sum\nolimits_{\mathbf{k}}u_3(0)\delta'(z)e^{i\mathbf{k}\mathbf{r}}\end{gathered} \tag{12.143}$$

for its component normal to the surface. The symbol $\delta'(z)$ denotes here the derivative of the δ-function with respect to the coordinate z. Making the Fourier transform with respect to the time, the contribution of the in-plane component of the gradient (equation (12.142)) to equation (12.106) becomes

$$\frac{q}{4\pi}\sum_{\mathbf{k}}\int_0^\infty dz'\int d\mathbf{r}'\frac{i\mathbf{k}\left(i\mathbf{k}\mathbf{v}+\frac{\partial u_3}{\partial z'}\right)}{\sqrt{(\mathbf{r}-\mathbf{r}')^2+(z-z')^2}}\cdot$$

$$\cdot e^{i\frac{\omega}{v_t}\sqrt{(\mathbf{r}-\mathbf{r}')^2+(z-z')^2}}e^{i\mathbf{k}\mathbf{r}'}+ \tag{12.144}$$

$$+\frac{q}{4\pi}\sum_{\mathbf{k}}\int d\mathbf{r}'\frac{i\mathbf{k}u_3(0)}{\sqrt{(\mathbf{r}-\mathbf{r}')^2+z^2}}e^{i\frac{\omega}{v_t}\sqrt{(\mathbf{r}-\mathbf{r}')^2+z^2}}e^{i\mathbf{k}\mathbf{r}'}\ .$$

Here, we introduce again the variable $\mathbf{r}_1=\mathbf{r}'-\mathbf{r}$ and use the result given by equation (12.138). Now, we can write the Fourier transform of $\mathbf{v}$ as given by equation (12.106) (including the contribution given by equation (12.140)) as

$$\widetilde{\mathbf{v}}(\mathbf{k}\omega;z)=-\frac{iq\mathbf{k}}{2\kappa_0}\int_0^\infty dz'\mathbf{k}\widetilde{\mathbf{v}}e^{i\kappa_0|z-z'|}-$$

$$-\frac{q\mathbf{k}}{2\kappa_0}\int_0^\infty dz'\frac{\partial\widetilde{u}_3}{\partial z'}e^{i\kappa_0|z-z'|}- \tag{12.145}$$

$$-\frac{q\mathbf{k}}{2\kappa_0}\widetilde{u}_3(0)e^{i\kappa_0 z}+\widetilde{\mathbf{F}}_{\parallel}(\mathbf{k}\omega;z)\ .$$

In the second integral in this equation we make an integration by parts and pass from $\partial/\partial z'$ to $-\partial/\partial z$ in the derivatives of function $e^{i\kappa_0|z-z'|}$. We get immediately the equation (12.109) given in the main text.

The gradient component normal to the surface (equation (12.143)) is treated in the same way. We introduce the Fourier transform with respect to time, then use equation (12.138) for the integration over $\mathbf{r}'$ and get the partial Fourier transform of the u_3. Thereafter, we perform an integration by parts in the first bracket in equation (12.143) which cancels out the contribution of the second bracket in this equation. Finally, we make another integration by parts for the term containing $\partial\widetilde{u}_3/\partial z'$ which cancels out the contribution of the δ'-term. We give here the contribution of this δ'-term, which is perhaps a bit more difficult to compute. We have successively

$$\int_{-\infty}^{+\infty}dz'\delta'(z')e^{i\kappa_0|z-z'|}=-\int_{-\infty}^{+\infty}dz'\delta(z')\frac{\partial}{\partial z'}e^{i\kappa_0|z-z'|}=$$
$$=\frac{\partial}{\partial z}\int_{-\infty}^{+\infty}dz'\delta(z')e^{i\kappa_0|z-z'|}=i\kappa_0 e^{i\kappa_0 z}\ . \tag{12.146}$$

This completes the proof of the derivation of the equations (12.109) and (12.110) given in the previous subsection.

12.4 P and S seismic waves

For seismic studies Earth can be viewed as an elastic continuous body, which is homogeneous and isotropic on the average, with a plane surface (half-space). Small deviations from homogeneity and isotropy can be viewed as small inhomogeneities, irregularities, distributed in the bulk and on the surface. The effect of the bulk inhomogeneities is a renormalization of the wave velocity and a small dispersion. This effect is estimated for plane waves, *i.e.* for the Fourier transforms of the waves. The dispersion transforms these plane waves in wave packets. The wave is retrieved by the inverse Fourier transform, which is slightly distorted by dispersion. The inverse Fourier transform produces interference. The surface inhomogeneities produce also scattered waves, which may lead to an appreciable distortion of the original waves. The primary P and S seismic waves produced by a seismic source localized in time and space are spherical-shell waves. On Earth's surface they produce the main shock, which has the more complex structure of a cylindrical-type wave.

The displacement of the P and S seismic waves is

$$\begin{aligned} \boldsymbol{u}_P &= -\tfrac{TM_4}{4\pi\rho c_l^3 R}\boldsymbol{n}\delta'(t - R/c_l) \ , \\ \boldsymbol{u}_S &= -\tfrac{T(M_4\boldsymbol{n}-\boldsymbol{M})}{4\pi\rho c_t^3 R}\delta'(t - R/c_t) \ , \end{aligned} \tag{12.147}$$

where T is the duration of the seismic activity in the focus, $\boldsymbol{n}$ is the unit vector from the focus to the observation point placed at distance R from the focus, ρ is the density of the Earth, $c_{l,t}$ are the propagation velocities of the longitudinal and transverse elastic waves and $M_i = M_{ij}n_j$, $M_4 = M_{ij}n_in_j$, where M_{ij} is the tensor of the seismic moment. Leaving aside the polarization, these waves have the generic form

$$u(t, R) = \frac{1}{R}\delta^{'}(ct - R) \ . \tag{12.148}$$

We compute here the Fourier transform of this function. Since

$$\begin{aligned} \delta(ct-R) &= \tfrac{1}{2\pi}\int dq e^{iq(ct-R)} \ , \\ \delta^{'}(ct-R) &= \tfrac{i}{2\pi}\int dq\cdot q e^{iq(ct-R)} \ , \end{aligned} \tag{12.149}$$

we get

$$u(t,R) = \frac{i}{2\pi}\int dq\cdot q e^{iqct}\frac{e^{-iqR}}{R} \ . \tag{12.150}$$

On the other hand,

$$\frac{e^{-iqR}}{R} = \frac{1}{(2\pi)^3}\int d\boldsymbol{k} f(\boldsymbol{k}) e^{i\boldsymbol{k}\boldsymbol{R}} \ , \tag{12.151}$$

where

$$f(\boldsymbol{k}) = \int d\boldsymbol{R}\frac{e^{-iqR}}{R}e^{-i\boldsymbol{k}\boldsymbol{R}} \ . \tag{12.152}$$

This integral is performed by introducing a cutoff factor $e^{-\mu r}$ and letting μ go to zero; we get

$$f(\boldsymbol{k}) = \frac{4\pi}{k^2-q^2} \tag{12.153}$$

and

$$u(t,R) = -\frac{2i}{(2\pi)^3}\int d\boldsymbol{k} e^{i\boldsymbol{k}\boldsymbol{R}}\int dq\cdot\frac{q e^{iqct}}{q^2-k^2} \ . \tag{12.154}$$

We can see that the Fourier transform $f(\boldsymbol{k})$ corresponds to free waves with frequency $\omega = qc$. In the integral over q we need to give a prescription for the singularities at $q = \pm k$. Since $u(t,R)$ should be vanishing for $t < 0$, according to the causality principle, the integration over q should be performed in the upper half-plane; therefore, the q-poles should lie in the upper half-plane, *i.e.* $q^2 - k^2$ should be viewed as $q^2 - k^2 - isgn(q)0^+$. We get

$$u(t,R) = \frac{1}{(2\pi)^2}\int d\boldsymbol{k} e^{i\boldsymbol{k}\boldsymbol{R}}(e^{ikct}+e^{-ikct}) \tag{12.155}$$

(for $t > 0$). Indeed, this integral is

$$u(t,R) = \frac{2}{\pi R}\int_0 dk\cdot k\sin kR\cos kct \ , \tag{12.156}$$

or

$$u(t,R) = -\frac{2}{\pi R}\frac{\partial}{\partial R}\int_0 dk \cos kR \cos kct \ , \qquad (12.157)$$

which gives

$$u(t,R) = -\frac{1}{R}\frac{\partial}{\partial R}\left[\delta(R-ct)+\delta(R+ct)\right] \ , \qquad (12.158)$$

i.e. the function given by equation (12.148) (for $t > 0$).
If we include the polarizations, the Fourier transform amounts to an integral of the form

$$\boldsymbol{g}(\boldsymbol{k}) = \int d\boldsymbol{R}\boldsymbol{n}\frac{e^{-iqR}}{R}e^{-i\boldsymbol{k}\boldsymbol{R}} \ . \qquad (12.159)$$

This integral can be performed by using

$$\int_{-\infty}^{+\infty} dx \frac{\sin px \cos qx}{x} = \begin{cases} \pi \ , \ p > q \ , \\ 0 \ , \ p < q \ . \end{cases} \qquad (12.160)$$

The result is

$$\boldsymbol{g}(\boldsymbol{k}) = \frac{4\pi\boldsymbol{k}}{k^3}\left[\frac{qk}{q^2-k^2} - \frac{i\pi}{2}\theta(k-q)\right] \ . \qquad (12.161)$$

13 Vibrations

13.1 Half-space

13.1.1 Introduction

According to empirical observations two faible tremors are felt on Earth's surface during an earthquake, followed by a wall-like main shock (or two main shocks).[1] The two tremors are localized in space and time and propagate with different velocities. They have been identified with the longitudinal P and the transverse S seismic waves. The earthquake produces elastic waves. On the other hand, the main shock(s), apart from an abrupt wall, which propagates with a slower velocity, exhibits a long tail. Obviously, it is a wave, but such a wave shape is unknown among the waves. The explaination of this curiosity, exhibited by any seismogram, is called the seismological problem, or Lamb's problem. It was felt that it is related to Earth's surface.

When Rayleigh discovered the damped "surface waves" on the free surface of an elastic half-space,[2] they have immediately been related to the seismic main shock. "It is not improbable that the surface waves here investigated play an important part in earthquakes, and in the collision of elastic solids" (Rayleigh, *loc. cit.*). Indeed, the Rayleigh surface waves propagate with a velocity smaller than the velocities of the elastic waves. Only that the Rayleigh surface waves are in

[1] R. D. Oldham, *Report on the Great Earthquake of 12th June, 1897*, Geol. Surv. India Memoir 29 (1899); "On the propagation of earthquake motion to long distances", Trans. Phil. Roy. Soc. London **A194** 135 (1900); C. G. Knott, *The Physics of Earthquake Phenomena*, Clarendon Press, Oxford (1908); A. E. H. Love, *Some Problems of Geodynamics*, Cambridge University Press, London (1911).

[2] Lord Rayleigh, "On waves propagated along the plane surface of an elastic solid", Proc. London Math. Soc. **17** 4 (1885) (J. W. Strutt, Baron Rayleigh, *Scientific Papers*, vol. 2, Cambridge University Press, London (1900), p. 441).

fact vibrations, while the seismic effects are waves. The difference between waves and vibrations is great: the waves have a wavefront which propagates, while the vibrations extend over the whole space, which oscillates.

The problem was taken up by Lamb,[3] who formulated it as a vibration problem for the Navier-Cauchy equation and a homogeneous and isotropic half-space with a plane free surface. The vibration problem requires an indefinite time and (spatial) boundary conditions on the surface. Lamb uses temporal Fourier transforms with respect to the frequency ω and in-plane Fourier tranforms with respect to the wavevector $\boldsymbol{k}$ parallel to the surface (though he gives main attention to a two-dimensional half-plane). He arrives at integrals with integrands having a denominator denoted Δ (see below) and including square roots like $\kappa_{1,2} = \sqrt{\omega^2/c_{1,2}^2 - k^2}$, where $c_{1,2}$ are the velocities of the elastic waves ($c_2 < c_1$).

Among many rather unphysical and irrelevant cases Lamb considers a force localized in space and, what is more important, localized in time (a time-impulse force $\sim \delta(t)$), suitable for earthquakes. These would be forced vibrations, *i.e.* vibrations produced by a force. This is the first inappropriate thing in Lamb's analysis. A force of the $\delta(t)$-type requires special conditions: for $t < 0$ we have no motion, the motion appears only after the force started to act, and exists for $t > 0$. This is the so-called causality principle, which says that any effect is subsequent to its cause, and the future is determined by the past. This principle and the causality condition it entails lead to waves, as expected for a force pertaining to earthquakes. We can see that the time is not indefinite in a propagating-wave problem, as it is in a vibration problem. A propagating-wave problem is completely distinct from a vibration problem. By treating a propagating-wave problem as a vibration problem, or viceversa, a vibration problem as a propagating-wave problem, is meaningless and leads to incorrect results. The contradiction resides in the fact that a vanishing integral, say $\int dx f(x) = 0$, does not imply necessarily $f(x) = 0$, while $f(x) = 0$ does imply a vanishing integral. The boundary conditions may mean $\int dx f(x) = 0$, but by using Fourier transforms we require $f(x) = 0$.

[3] H. Lamb, "On the propagation of tremors over the surface of an elastic solid", Phil. Trans. Roy. Soc. (London) **A203** 1 (1904).

The problem becomes especially dangerous for discontinuous, singular functions like δ-functions. For instance, $\delta(x)$ is vanishing for any $x \neq 0$, but the waves e^{ikx} in its Fourier transform $\delta(x) = \frac{1}{2\pi}\int dke^{ikx}$ are nowhere vanishing.

Further on, Lamb identifies in his integrals two types of contributions. One arises from the zeros of the denominator Δ (the residues), which corresponds to Rayleigh surface-wave contribution. The corresponding contribution is identified by Lamb as the seismic main shock. Another contribution, Lamb claims, appears from the cuts in the complex plane associated to the zeros of the square roots $\kappa_{1,2} = \sqrt{\omega^2/c_{1,2}^2 - k^2}$ (branch points). According to Lamb, these cuts start from $\omega = \pm c_{1,2}k$ and go to infinity, such that a contour at infinity would cross such lines, and would give non-uniform functions, which make the integration result indefinite. Therefore, we should go along the cuts, such as to circumvent them, a procedure which gives additional contributions to the integrals. Surprisingly, such a procedure gives waves, claims Lamb, which he assigns to the seismic P and S waves. So, it is claimed that Lamb was the first to explain the seismograms, and solved thus the seismological problem. However, it is curious that a vibration problem gives waves, isn't it? Unfortunately, the cuts for $\kappa_{1,2} = \sqrt{\omega^2/c_{1,2}^2 - k^2}$ are not from the branch points $\omega = \pm c_{1,2}k$ to infinity, but from $-c_2k$ to c_2k, and from $-\infty$ to $-c_1k$ and from c_1k to $+\infty$ (with the point at infinity as a single point). A cut from ck to infinity appears for $\sqrt{\omega/c - k}$, but not for $\sqrt{\omega^2/c^2 - k^2}$. The correct cuts bring no contribution to Lamb's integrals, so he did not get the seismic P and S waves, as expected from a vibration treatment.

The full, exact evaluation of Lamb's integrals has never been accomplished, in spite of numerical calculations and the use of Laplace transforms.[4] However, the Rayleigh surface-wave contribution for a temporal impulse (the residues arising from the zeros of Δ, associated by Lamb to the main shock) can straightforwardly be qualitatively

[4] L. Cagniard, *Reflection and Refraction of Progressive Seismic Waves* (translated by E. A. Flinn and C. H. Dix), McGraw-Hill, NY (1962); A. T. de Hoop, *Representation theorems for the displacement in an elastic solid and their applications to elastodynamic diffraction theory*, D. Sc. Thesis, Technische Hogeschool, Delft (1958); "Modification of Cagniard's method for solving seismic pulse problems", Appl. Sci. Res. **B8** 349 (1960).

estimated (see below). It looks like a displacement extended over the whole free surface, valid at any moment, with a scissor-like (double-wall) bump which propagates with velocity $c < c_2$. The infinite spatial extension and non-vanishing values for any time are specific to vibrations, the propagating bump is specific to waves, coming from a $\delta(t)$-source. After a long time, this motion extincts. This solution is very different from the seismic main shock, which is an abrupt wall, with a long tail, propagating with the velocity $c_{1,2}$, and vanishing beyond the wall position. The seismic P and S waves propagate on Earth's surface with velocities larger than $c_{1,2}$, which is easily seen from the projection of their wavefronts on Earth's surface.

We summarize here briefly the theory of the seismic P and S waves and the seismic main shock(s).[5] First, the seismic focus is localized in space and time at a depth below the Earth's surface much smaller than the radius of the Earth. Therefore, we may assume, for distances of our interest, a half-space for the Earth, which we take homogeneous and isotropic. Next, we have established the seismic tensorial point force acting in the focus, governed by the seismic moment (as for a shear faulting; an isotropic moment corresponds to explosions). The static deformations produced by such a (static) force have been computed.[6] This force produces two waves, which look like spherical shells and propagate with velocities $c_{1,2}$. These are the seismic P and S waves. When arrived at the Earth's surface, the crossing circles of the wavefronts with the surface propagate (their radii increase) with higher velocities than $c_{1,2}$. Along these circles a reaction force acts on behalf of the surface (boundary force), which generates secondary waves, according to Huygens principle. The secondary waves are in fact the reflected waves of the singular primary waves. On Earth's surface the secondary waves look like two walls (for different components of the displacement), with abrupt wavefronts propagating with velocities $c_{1,2}$, exhibiting long tails. These are the seismic main

[5]B. F. Apostol, "Elastic waves inside and on the surface of a half-space", Quart. J. Mech. Appl. Math. **70** 289 (2017); *Introduction to the Theory of Earthquakes*, Cambridge International Science Publishing, Cambridge (2017); *The Theory of Earthquakes*, Cambridge International Science Publishing, Cambridge (2017); *Seismology*, Nova, NY (2020).

[6]B. F. Apostol, "Elastic displacement in a half-space under the action of a tensor force. General solution for the half-space with point forces", J. Elas. **126** 231 (2017).

shocks. The crossing circles of the primary seismic P and S waves leave behind seismic main shocks.

The process of wave generation and propagation described above for an earthquake is a transient process, in the sense that the seismic primary waves and the wall of the seismic main shock come and go to infinity in time, and the coda (the tail) of the main shock subsides progressively to zero. Meantime, the force in the seismic focus has ceased since long. Obviously, boundary conditions on Earth's surface are meaningless, since this motion does not affect all the time the entire surface. A vibration problem is improper in this case. However, after multiple reflections on the Earth's spherical surface the motion may embrace the whole Earth's surface and will lasts indefinitely, without no force. These are the free oscillations (eigenvibrations, normal modes) of the Earth, viewed either a a half-space or as a sphere.

Nevertheless, for the sake of completeness and for its historical interest in seismology we may formulate a vibration problem with a force (forced vibrations) in these cases.

13.1.2 Lamb's problem

The vibration problem of a homogeneous and isotropic elastic half-space (Lamb's problem) is a long-standing problem.[7] Traditionally, it is related to the seismic waves and the seismic main shock propagating during earthquakes on Earth's surface. Fourier and Laplace transform techniques have been employed to solve this problem.[8] Though the main feature - the Rayleigh surface waves- is known since long,[9]

[7] H. Lamb, "On the propagation of tremors over the surface of an elastic solid", Phil. Trans. Roy. Soc. London **A203** 1 (1904); K. Aki and P. G. Richards, *Quantitative Seismology*, University Science Books, Sausalito, CA (2009); A. Ben-Menahem and J. D. Singh, *Seismic Waves and Sources*, Springer, New York (1981).

[8] L. Cagniard, *Reflection and Refraction of Progressive Seismic Waves* (translated by E. A. Flinn and C. H. Dix), McGraw-Hill, NY (1962); A. T. de Hoop, *Representation theorems for the displacement in an elastic solid and their applications to elastodynamic diffraction theory*, D. Sc. Thesis, Technische Hogeschool, Delft (1958); "Modification of Cagniard's method for solving seismic pulse problems", Appl. Sci. Res. **B8** 349 (1960).

[9] Lord Rayeigh, "On waves propagated along the plane surface of an elastic solid", Proc. London Math. Soc. **17** 4 (1885) (J. W. Strutt, Baron Rayleigh, *Scientific*

the progress towards an explicit full solution requires, on one side, a physically sound wave-vibration force source and, on the other, a consistent differentiation between vibrations and propagating waves. Recently, a general, formal scheme of solution has been analyzed and the difference between waves and vibrations has been emphasized.[10]

The seismic tensorial point force governed by the seismic moment tensor has been introduced.[11] The static deformations produced by this force in a homogeneous and isotropic elastic half-space have been computed and the waves generated in a homogeneous and isotropic body have been derived for a temporal-impulse seismic moment. The seismic P and S waves have been obtained and the main shock generated by these waves on the free surface of the half-space has been computed. The seismic waves are spherical shells (proportional to the derivative of the δ-function) and the main shock has the shape of an abrupt wall propagating on the surface, vanishing beyond the position of the wall (actually, there exist two main shocks, for different components of the displacement). We present herein the full explicit solution of the vibration problem with the (isotropic) tensorial point force in a homogeneous and isotropic half-space with a (free) plane surface. The solution is obtained by introducing vector plane-wave functions.

The seismic focus is localized over distances much smaller than our scale distances; therefore, a point focal force is justified. The seismic activity in an earhquake focus lasts a short time in comparison to our scale time; therefore, a temporal-impulse of the focal force is also justified. Moreover, the depth of earthquake foci is very small in comparison to Earth's radius; therefore, the approximation of a half-space for the seismic effects is justified, at least for not very long propagation times. Under these conditions the seismic waves are localized (singular) and obey the causality principle, *i.e.* they have the shape of propagating spherical shells which appear only after the

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Papers, vol. **2**, pp. 441-447. Cambridge University Press, London (1900)).

[10]B. F. Apostol, "On the Lamb problem: forced vibrations in a homogeneous and isotropic elastic half-space", Arch. Appl. Mech. **90** 2335 (2020).

[11]B. F. Apostol, "Elastic displacement in a half-space under the action of a tensor force. General solution for the half-space with point forces", J. Elast. **126** 231 (2017); "Elastic waves inside and on the surface of an elastic half-space", Quart. J. Mech. Appl. Math. **70** 281 (2017).

seismic activity started in the focus; they come to any location and pass over, rather quickly. Once arrived at the surface of the half-space these waves generate secondary waves, according to Huygens principle, which, on the surface, propagates as an abrupt wall-like main shock (actually, two main shocks). A vibration problem, which requires necessarily boundary conditions, is obviously a different problem, since the waves are not present permanently on the entire surface, and, for the main shock, the surface is the location of the wave sources. The wave propagation in these conditions is a transient phenomenon, not a stationary one.

For the vibration problem we consider a wave source which lasts an indefinite time. Such a source may generate delocalized waves, which propagate both in the future and in the past. These waves may obey boundary conditions. This is a valid vibration problem (of forced vibrations), which we solve herein for a half-space. The Rayleigh surface waves bring the main contribution to the solution (for a free surface). In the special limit of a temporal-impulse source the solution extends over the entire free surface, exhibiting a (scissor-like) double-wall, whih propagates both in the future and in the past. This solution is different from the abrupt one-wall seismic main shock. After the temporal-impulse seismic force ceases its action we may have free oscillations (vibrations), which, for the half-space, are governed by the Rayleigh surface wave frequency.

13.1.3 Vibration equation

The elastic vibrations of a homogeneous and isotropic body are described by the Navier-Cauchy equation

$$c_2^2 curl\, curl\mathbf{u} - c_1^2 grad\, div\mathbf{u} - \omega^2\mathbf{u} = \mathbf{F} \ , \tag{13.1}$$

where $\boldsymbol{u}$ is the time Fourier transform of the local displacement, $c_{1,2}$ are the velocities of the elastic waves, ω is the frequency and $\boldsymbol{F}$ is the time Fourier transform of the force (per unit mass).[12] We consider this equation in a half-space $z < 0$, with a free, or fixed, plane surface

[12] L. Landau and E. Lifshitz, *Course of Theoretical Physics*, vol. 7, *Theory of Elasticity*, Elsevier, Oxford (1986).

$z=0$, for the seismic tensorial force with components

$$F_i = m_{ij}\partial_j\delta(\boldsymbol{R}-\boldsymbol{R}_0) \ , \tag{13.2}$$

placed at $\boldsymbol{R}_0=(0,0,z_0)$, $z_0\leq 0$, where m_{ij} are the cartesian components of the seismic (symmetric) tensor ($i,j=x,y,z$). The position vector is $\boldsymbol{R}=(\boldsymbol{r},z)$, where $\boldsymbol{r}=(x,y)$ is the in-plane position vector (parallel to the surface $z=0$), with horizontal coordinates x, y, and z is the perpendicular-to-surface coordinate (vertical coordinate, $z<0$). In these equations the unknown function $\boldsymbol{u}$ depends on ω and $\boldsymbol{R}$ ($\boldsymbol{u}=\boldsymbol{u}(\omega;\boldsymbol{R})$) and the seismic tensor depends on ω ($m_{ij}=m_{ij}(\omega)$). For simplicity, we omit occasionally the position-coordinates variables and the arguments of the Fourier transforms, which can be read easily from the context.

We introduce the orthogonal vector plane waves

$$\begin{gathered}\boldsymbol{Z}(\boldsymbol{k};\boldsymbol{r})=\boldsymbol{e}_z\frac{e^{i\boldsymbol{kr}}}{2\pi} \ , \\ \boldsymbol{G}(\boldsymbol{k};\boldsymbol{r})=\frac{i}{k}(k_x\boldsymbol{e}_x+k_y\boldsymbol{e}_y)\frac{e^{i\boldsymbol{kr}}}{2\pi} \ , \\ \boldsymbol{C}(\boldsymbol{k};\boldsymbol{r})=\frac{i}{k}(k_y\boldsymbol{e}_x-k_x\boldsymbol{e}_y)\frac{e^{i\boldsymbol{kr}}}{2\pi} \ , \end{gathered} \tag{13.3}$$

where $\boldsymbol{k}$ is the in-plane wavevector and $\boldsymbol{e}_i$, $i=x,y,z$, are the unit vectors along the x,y,z-directions, and use the decompositions

$$\begin{gathered}\boldsymbol{u}(\omega,\boldsymbol{R})=\int d\boldsymbol{k}[f(\omega,\boldsymbol{k};z)\boldsymbol{Z}(\boldsymbol{k},\boldsymbol{r})+ \\ +g(\omega,\boldsymbol{k};z)\boldsymbol{G}(\boldsymbol{k},\boldsymbol{r})+h(\omega,\boldsymbol{k};z)\boldsymbol{C}(\boldsymbol{k},\boldsymbol{r})] \ ,\end{gathered} \tag{13.4}$$

and

$$\begin{gathered}\boldsymbol{F}(\omega,\boldsymbol{R})=\int d\boldsymbol{k}[F_z(\omega,\boldsymbol{k};z)\boldsymbol{Z}(\boldsymbol{k},\boldsymbol{r})+ \\ +F_g(\omega,\boldsymbol{k};z)\boldsymbol{G}(\boldsymbol{k},\boldsymbol{r})+F_c(\omega,\boldsymbol{k};z)\boldsymbol{C}(\boldsymbol{k},\boldsymbol{r})] \ .\end{gathered} \tag{13.5}$$

The function $\boldsymbol{G}$ is the gradient of a plane wave ($\boldsymbol{G}$ from "gradient") and the function $\boldsymbol{C}$ is the vertical component of the *curl* of a plane wave ($\boldsymbol{C}$ from "*curl*"). These functions are constructed following the example of the vector spherical and cylindrical harmonics.[13] Equation (13.1) becomes

$$\begin{aligned} &c_1^2 f'' + (\omega^2 - c_2^2 k^2) f - (c_1^2 - c_2^2) k g' = -F_z \ , \\ &c_2^2 g'' + (\omega^2 - c_1^2 k^2) g + (c_1^2 - c_2^2) k f' = -F_g \ , \\ &c_2^2 h'' + (\omega^2 - c_2^2 k^2) h = -F_c \ , \end{aligned} \tag{13.6}$$

where the derivatives are taken with repect to the variable z and

$$\begin{aligned} &F_z = \tfrac{i}{2\pi} m_{z\alpha} k_\alpha \delta(z - z_0) + \tfrac{1}{2\pi} m_{zz} \delta'(z - z_0) \ , \\ &F_g = \tfrac{1}{2\pi k} m_{\alpha\beta} k_\alpha k_\beta \delta(z - z_0) - \tfrac{i}{2\pi k} m_{z\alpha} k_\alpha \delta'(z - z_0) \ , \\ &F_c = \tfrac{1}{2\pi k} (m_{x\alpha} k_\alpha k_y - m_{y\alpha} k_\alpha k_x) \delta(z - z_0) - \\ &- \tfrac{i}{2\pi k} (m_{xz} k_y - m_{yz} k_x) \delta'(z - z_0) \ , \end{aligned} \tag{13.7}$$

where $\alpha, \beta = x, y$. An anisotropic seismic moment is specific to earthquakes which produce seismic waves (by a shear faulting). Although the solution can be obtained for this general form, for vibrations we prefer an isotropic tensor $m_{ij} = -m\delta_{ij}$, which simplifies greatly the calculations. The salient features of the solution are not affected by this simplification. For an isotropic seismic moment the force components become

$$\begin{aligned} &F_z = -\tfrac{m}{2\pi} \delta'(z - z_0) \ , \ F_g = -\tfrac{mk}{2\pi} \delta(z - z_0) \ , \\ &F_c = 0 \ . \end{aligned} \tag{13.8}$$

An average over the directions of the wavevector $\boldsymbol{k}$ in equations (13.7) leads to the same force components (with different coefficients).

[13] H. Lamb, "On the vibrations of an elastic sphere", Proc. London Math. Soc. **13** 189 (1882); "On the oscillations of a viscous spheroid", Proc. London Math. Soc. **13** 51 (1881); W. W. Hansen, "A new type of expansion in radiation problems", Phys. Rev. **47** 139 (1935); R. G. Barrera, G. A. Estevez and J. Giraldo, "Vector spherical harmonics and their applications to magnetostatics", Eur. J. Phys. **6** 287 (1985); W. R. Smythe, *Static and Dynamic Electricity*, McGraw-Hill (1950).

The boundary conditions are $\sigma_{iz}\mid_{z=0}=P_i$, where $\sigma_{ij}=c_2^2u_{ij}+(c_1^2-2c_2^2)div\boldsymbol{u}\cdot\delta_{ij}$ is the stress tensor, $u_{ij}=\frac{1}{2}(\partial_iu_j+\partial_ju_i)$ is the strain tensor and P_i are the components of the force acting upon the surface (force per unit cross-section area divided by the density of the body).[14] For a free surface ($P_i=0$) we get the boundary conditions

$$(g'+kf)\mid_{z=0}=0\ ,\ \left[c_1^2f'-(c_1^2-2c_2^2)kg\right]_{z=0}=0\ ,$$
$$h'\mid_{z=0}=0\ ; \tag{13.9}$$

for a fixed surface the boundary conditions are $f,g,h\mid_{z=0}=0$ ($u\mid_{z=0}=0$). For $F_c=0$ and a free or fixed surface the solution of the third equation (13.8) is $h=0$. We are left with the equations

$$c_1^2f''+c_2^2\kappa_2^2f-(c_1^2-c_2^2)kg'=\tfrac{m}{2\pi}\delta'(z-z_0)\ ,$$
$$c_2^2g''+c_1^2\kappa_1^2g+(c_1^2-c_2^2)kf'=\tfrac{mk}{2\pi}\delta(z-z_0)\ , \tag{13.10}$$

where $\kappa_{1,2}^2=\omega^2/c_{1,2}^2-k^2$. The system of homogeneous equations (13.10) has the eigenvalues $\kappa_{1,2}^2$; therefore, the free solution is

$$f_0=A\kappa_1e^{-i\kappa_1z}+iBke^{-i\kappa_2z}+c.c\ ,$$
$$g_0=iAke^{-i\kappa_1z}+B\kappa_2e^{-i\kappa_2z}+c.c\ , \tag{13.11}$$

where the (complex) coefficients A and B are determined by the boundary conditions ($\kappa_{1,2}\neq0$). We use the convention of replacing $\kappa_{1,2}=\sqrt{\omega^2/c_{1,2}^2-k^2}$, $\omega^2/c_{1,2}^2-k^2>0$, by $i\kappa_{1,2}$ for $\kappa_{1,2}^2=k^2-\omega^2/c_{1,2}^2>0$. A particular solution of equations (13.10) is obtained by Fourier transforming these equations with respect to the variable z; we get the particular solution

$$f_p=sgn(z-z_0)\tfrac{m}{4c_1^2}e^{i\kappa_1|z-z_0|}+c.c\ ,$$
$$g_p=-i\tfrac{mk}{4\kappa_1c_1^2}e^{i\kappa_1|z-z_0|}+c.c\ . \tag{13.12}$$

[14]L. Landau and E. Lifshitz, *Course of Theoretical Physics*, vol. 7, *Theory of Elasticity*, Elsevier, Oxford (1986).

Finally, by making use of the boundary conditions for the free surface (equations (13.9)), we get the full solution

$$f = \frac{(\kappa_2^2-k^2)^2-4\kappa_1\kappa_2k^2}{\Delta}\frac{m}{4c_1^2}e^{-i\kappa_1(z+z_0)}+$$
$$+\frac{(\kappa_2^2-k^2)k^2}{\Delta}\frac{m}{c_1^2}e^{-i\kappa_1z_0-i\kappa_2z}+ \tag{13.13}$$
$$+sgn(z-z_0)\frac{m}{4c_1^2}e^{i\kappa_1|z-z_0|}+c.c$$

and

$$g = \frac{i\left[(\kappa_2^2-k^2)^2-4\kappa_1\kappa_2k^2\right]k}{\kappa_1\Delta}\frac{m}{4c_1^2}e^{-i\kappa_1(z+z_0)}-$$
$$-i\frac{(\kappa_2^2-k^2)\kappa_2k}{\Delta}\frac{m}{c_1^2}e^{-i\kappa_1z_0-i\kappa_2z}- \tag{13.14}$$
$$-i\frac{mk}{4\kappa_1c_1^2}e^{i\kappa_1|z-z_0|}+c.c\ ,$$

where

$$\Delta = (\kappa_2^2-k^2)^2+4\kappa_1\kappa_2k^2 \tag{13.15}$$

is the determinant of the system of equations arising from the boundary conditions. For a fixed surface the solution is

$$\begin{gathered}
f = \frac{k^2-\kappa_1\kappa_2}{k^2+\kappa_1\kappa_2}\frac{m}{4c_1^2}e^{-i\kappa_1(z+z_0)}- \\
-\frac{k^2}{k^2+\kappa_1\kappa_2}\frac{m}{2c_1^2}e^{-i\kappa_1z_0-i\kappa_2z}+ \\
+sgn(z-z_0)\frac{m}{4c_1^2}e^{i\kappa_1|z-z_0|}+c.c\ , \\
g = \frac{i(k^2-\kappa_1\kappa_2)k}{\kappa_1(k^2+\kappa_1\kappa_2)}\frac{m}{4c_1^2}e^{-i\kappa_1(z+z_0)}+ \\
+i\frac{\kappa_2k}{k^2+\kappa_1\kappa_2}\frac{m}{2c_1^2}e^{-i\kappa_1z_0-i\kappa_2z}- \\
-i\frac{mk}{4\kappa_1c_1^2}e^{i\kappa_1|z-z_0|}+c.c\ .
\end{gathered} \tag{13.16}$$

The above formulae give the functions $f(\omega,k;z)$ and $g(\omega,k;z)$. By inserting these functions in the first equation (13.4) and performing the $\boldsymbol{k}$-integrations we get the solution $\boldsymbol{u}(\omega;\boldsymbol{R})$; it can be represented as

$$\begin{gathered}\boldsymbol{u}(\omega;\boldsymbol{R})=\boldsymbol{e}_z\int dkkf(\omega,k;z)J_0(kr)-\\ -\boldsymbol{e}_r\int dkkg(\omega,k;z)J_1(kr)+c.c.\ ,\end{gathered}\tag{13.17}$$

where $J_{0,1}(kr)$ are Bessel functions and $\boldsymbol{e}_r$ is the radial unit vector. Finally, a frequency Fourier transform gives the desired solution $\boldsymbol{u}(t,\boldsymbol{R})$. This completes formally the solution of the forced vibration problem.[15] We may perform first the ω-integration. If the force is a harmonic oscillation with frequency ω_0, *i.e.* if $m(t)=m\cos\omega_0 t$ ($m(\omega)\sim\pi m\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$, the solution looks like

$$\begin{gathered}\boldsymbol{u}(t,\boldsymbol{R})=[\boldsymbol{e}_z\int dkkf(k;z)J_0(kr)-\\ -\boldsymbol{e}_r\int dkkg(k;z)J_1(kr)]\cos\omega_0 t+c.c.\ ,\end{gathered}\tag{13.18}$$

where $f(k;z)=f(\omega_0,k;z)$ and $g(k;z)=g(\omega_0,k;z)$ are analytic functions and $m(\omega_0)$ is replaced by m. If the functions $f(\omega,k;z)$ and $g(\omega,k;z)$ have poles at some frequency ω_s, then contributions of the form $\sim\delta(\omega_s\pm\omega_0)$ may appear, which indicate resonances (for $\omega_0=\pm\omega_s$). The solution given by equation (13.18) has the typical form of a vibration, with the time dependence separated from the position dependence.

The description given above can also be applied to a two-dimensional space (a half-plane), defined by the coordinates $\boldsymbol{r}=(x,z)$, $z<0$, with the source placed at $x=0$, $z=z_0\leq 0$. The vector functions given by equations (13.3) become $\boldsymbol{Z}(k;x)=\boldsymbol{e}_z\frac{e^{ikx}}{\sqrt{2\pi}}$ and $\boldsymbol{G}(k;x)=i\boldsymbol{e}_x\frac{e^{ikx}}{\sqrt{2\pi}}$ (the function $\boldsymbol{C}(k;x)$ is irrelevant). The displacement is given by

$$\begin{gathered}\boldsymbol{u}(\omega;\boldsymbol{r})=\boldsymbol{e}_z\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}dkf(\omega,k;z)e^{ikx}+\\ \boldsymbol{e}_x\frac{i}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}dkg(\omega,k;z)e^{ikx}+c.c.\end{gathered}\tag{13.19}$$

(a result which can be obtained straightforwardly from equation (13.17) by integrating over y).

[15] It is an old saying that "a problem is solved when it is reduced to quadratures".

13.1.4 Surface-wave contribution

Let us perform first the ω-integration in equations (13.17), and assume a smooth function $m(\omega)$. The main contribution to this integration comes from the zeros of the denominators in equations (13.14) and (13.16), if they exist. The branch lines associated to $\kappa_{1,2}$ are from $-c_2k$ to c_2k, from $-\infty$ to $-c_1k$ and from c_1k to $+\infty$; they bring no special contribution. For a fixed surface the denominators in equations (13.16) do not vanish ($\kappa_1 \neq 0$). The so-called "lateral waves" associated to $\kappa_{1,2} = 0$, *i.e.* waves which do not depend on the coordinate z, are not produced by a source, so the boundary conditions cannot be satisfied and the solution remains undeterminate. For a free surface the determinant Δ (equation (13.15)) is vanishing for only two acceptable frequencies given by $\omega_s = \pm c_2\xi_0 k$, where ξ_0 varies between 0.87 and 0.95 (it depends on the ratio c_2/c_1, which varies between $1/\sqrt{2}$ and 0).[16] We note that this solution corresponds to damped waves ($\kappa_{1,2} \rightarrow i\kappa_{1,2}$). These are the well-known Rayleigh surface waves. We need to compute the residues of the ω-integration of the functions f and g given by equations (13.14). To this end, we expand the determinant Δ in the vicinity of ω_s,

$$\Delta \simeq \frac{k^2}{c_2^2\sqrt{1-\xi_0^2}}(\omega^2 - \omega_s^2) \ . \tag{13.20}$$

Here and in all the subsequent computations we use the (numerical) approximations $\xi_0 \simeq 1$ and $c_2^2/c_1^2 \ll 1$, wherever appropriate. In addition, we limit ourselves to surface vibrations ($z = 0$). We get for the residues of the functions f and g

$$\begin{gathered} f_s \simeq g_s/2\sqrt{1-\xi_0^2} \simeq \\ \simeq \sqrt{1-\xi_0^2}\tfrac{c_2k}{2c_1^2}sgn(t)Im\left[m(ck)e^{ickt}\right]e^{-k|z_0|} \ , \end{gathered} \tag{13.21}$$

where $c = c_2\xi_0$ ($\omega_s = ck$). For a harmonic oscillation $m(ck) \sim \delta(ck \pm \omega_0)$ with frequency ω_0, we get the vertical (z) and the radial (horizontal, r) surface displacements $u_{z,r} \sim \sin\omega_0 \mid t \mid e^{-\omega_0|z_0|/c}J_{0,1}(\omega_0 r/c)$,

[16] L. Landau and E. Lifshitz, *Course of Theoretical Physics*, vol. 7, *Theory of Elasticity*, Elsevier, Oxford (1986); B. F. Apostol, "On the Lamb problem: forced vibrations in a homogeneous and isotropic elastic half-space", Arch. Appl. Mech. **90** 2335 (2020).

according to equations (13.18). It is worth noting that equation (13.21) includes waves propagating in both time directions, *i.e.* it has an acausal structure, specific to vibrations.

If the force is absent the boundary conditions (equations (13.9)) give a homogeneous system of equations for the coefficients A and B of the free solutions (equations (13.11)), which leads to $A, B \sim \delta(\omega^2 - \omega_s^2)$ (arising from $\Delta(A, B) \sim (\omega^2 - \omega_s^2)(A, B) = 0$); the solutions are free oscillations (vibrations, normal modes) proportional to $\cos \omega_s t$, $\sin \omega_s t$, whose spatial dependence remains undetermined (as expected).

Now we examine the special case of a temporal-impulse source $m(t) = mT\delta(t)$, where T is the short duration of the impulse. This is an improper case, since a temporal-impulse source requires causal conditions, while the vibration solution obtained above is acausal. The result which is obtained below for this special case is unphysical. However, we include it here for the sake of completeness and for the interest it may arouse in regard to the seismic waves. The main contribution to the vertical displacement (f_s) becomes

$$u_z \simeq \sqrt{1 - \xi_0^2} \frac{mc_2 T}{2c_1^2} \int_0^\infty dk k^2 \sin ck \mid t \mid e^{-k|z_0|} J_0(kr) \ . \qquad (13.22)$$

By analytic continuation of the Weyl-Sommerfeld integrals,[17] we get

$$\begin{aligned} \int_0^\infty dk e^{-k(|z_0| - ic|t|)} J_0(kr) &= \frac{1}{\sqrt{r^2 + (|z_0| - ic|t|)^2}} = \\ &= \left[(R_0^2 - c^2 t^2)^2 + 4c^2 z_0^2 t^2 \right]^{-1/4} e^{i\chi/2} \ , \end{aligned} \qquad (13.23)$$

where $R_0^2 = r^2 + z_0^2$ and

$$\tan \chi = \frac{2c \mid z_0 t \mid}{R_0^2 - c^2 t^2} \qquad (13.24)$$

($z_0 \neq 0$). The vertical displacement becomes

$$u_z \simeq -\sqrt{1 - \xi_0^2} \frac{mT}{2c_2 c_1^2} \frac{\partial^2}{\partial t^2} I(t, r) \ , \qquad (13.25)$$

[17] A. Sommerfeld, *Partielle Differentialgleichungen der Physik. Vorlesungen ueber Theoretische Physik*, Bd. VI. Akad. Verlag. Leipzig (1966).

where

$$I(t,r) = \left[(R_0^2 - c^2t^2)^2 + 4c^2z_0^2t^2\right]^{-1/4} \sin\frac{\chi}{2} \ . \tag{13.26}$$

The function $I(t,r)$ is vanishing for $r \to \infty$ and any time, and for $\mid t \mid \to \infty$ and any position; it has a (scissor-like) double-wall behaviour around $R_0 = c \mid t \mid$; this feature propagates on the surface with velocity c. The propagating double-wall comes from the surface waves and the temporal-impulse $m(t) = mT\delta(t)$. The infinite spatial extension of this solution and its dependence on $\mid t \mid$ are specific to vibrations, while its propagating character reflects waves. This dual, hybrid result, arising from treating a propagating-wave problem as a vibration problem, is different from the seismic main shock, as shown by seismograms, which has a causal nature and exhibits an abrupt wall-like structure, vanishing for distances beyond the wall position. For $z_0 = 0$ the displacement on the surface is zero.

As a consequence of the sharp jump of the angle χ from $\pi/2$ to $-\pi/2$ (equation (13.24)) in the vicinity of $R_0 = c \mid t \mid$, the function $I(t,r)$ can be represented in this vicinity as $I = \frac{\sqrt{2}}{2}\left(2R_0 \mid z_0 \mid\right)^{-1/2} sgn(R_0 - c \mid t \mid)$, such that the vertical displacement is represented as

$$u_z \simeq -\sqrt{1-\xi_0^2}\frac{mc_2T}{2c_1^2}\frac{1}{\sqrt{R_0 \mid z_0 \mid}}\delta'(R_0 - c \mid t \mid) \ . \tag{13.27}$$

A similar representation is valid for the main contribution to the radial (horizontal) displacement

$$u_g \simeq -\left(1-\xi_0^2\right)\frac{mc_2T}{c_1^2 r}\sqrt{\frac{\mid z_0 \mid}{R_0}}\delta'(R_0 - c \mid t \mid) \ . \tag{13.28}$$

As expected, these displacements are very different from the propagating waves generated by a temporal-impulse, which go like $\sim \delta'(R_0 - c_{1,2}t)/R_0$. A similar calculation for a half-plane (according to equations (13.19)) leads to two bumps for the displacement components, placed at $x = \pm\sqrt{z_0^2 + c^2t^2}$ and propagating with velocities $\pm c$. These solutions are very different from the propagating waves in two dimensions, generated by a temporal impulse, which are divergent (proportional to $\delta(r - c_{1,2}t)/\sqrt{c_{1,2}^2t^2 - r^2}$). Finally, we note that the decomposition in vector plane waves given by equation (13.4) can also be used for the propagating-wave problem. It can be checked easily that this method leads straightforwardly to the spherical-shell waves generated by the temporal-impulse tensorial point force. Also, the vector plane waves can be used for other vibration problems with cylindrical geometry, as the vibrations of a plane-parallel slab[18] or the vibrations at the plane interface of two solids.[19]

13.1.5 Waves

By using the vector plane-waves functions the Navier-Cauchy equations (13.6) for the force given by equation (13.8) become

$$\begin{gathered}\frac{\partial^2 f}{\partial t^2} - c_1^2 f'' + c_2^2 k^2 f + (c_1^2 - c_2^2) k g' = -\frac{m}{2\pi}\delta'(z - z_0) \ , \\ \frac{\partial^2 g}{\partial t^2} - c_2^2 g'' + c_1^2 k^2 g - (c_1^2 - c_2^2) k f' = -\frac{mk}{2\pi}\delta(z - z_0) \ , \\ \frac{\partial^2 h}{\partial t^2} - c_2^2 h'' + c_2^2 k^2 h = 0 \ ,\end{gathered} \tag{13.29}$$

where the time is restored and m stands for $m(t) = mT\delta(t)$. We represent the functions in these equations as Fourier integrals with e^{iqz}, and retain the restriction of these functions to the half-space $z < 0$. The third equation (13.29) represents free waves, which we are not interested in. The remaining two equations are

$$\begin{gathered}\frac{\partial^2 f}{\partial t^2} + (c_1^2 q^2 + c_2^2 k^2) f + i(c_1^2 - c_2^2) qkg = -\frac{imq}{2\pi} e^{-iqz_0} \ , \\ \frac{\partial^2 g}{\partial t^2} + (c_2^2 q^2 + c_1^2 k^2) g - i(c_1^2 - c_2^2) qkf = -\frac{mk}{2\pi} e^{-iqz_0} \ .\end{gathered} \tag{13.30}$$

In these equations we perform a time Fourier transform, with frequency ω, which according to the causality principle should have a small negative imaginary part ($\omega \rightarrow \omega + i\varepsilon$, $\varepsilon \rightarrow 0^+$). Indeed, in the ω-integration $\int d\omega e^{-i\omega t}...$ this condition leads to 0 for $t < 0$ (integration over upper half-plane) and to a non-vanishing result for $t > 0$ (integration over the lower half-plane, where the poles exist, $\omega = ... - i\omega$).

[18] A. E. H. Love, *Some Problems of Geodynamics*, Cambridge University Press, London (1911) (Dover, NY (1967).

[19] R. Stoneley, "Elastic waves at the surface of separation of two solids", Proc. Roy. Soc. London **A106** 416 (1924).

Equations (13.30) become

$$\begin{aligned}(c_1^2q^2+c_2^2k^2-\omega^2)f+i(c_1^2-c_2^2)qkg&=-\tfrac{imTq}{2\pi}e^{-iqz_0}\ ,\\ (c_2^2q^2+c_1^2k^2-\omega^2)g-i(c_1^2-c_2^2)qkf&=-\tfrac{mTk}{2\pi}e^{-iqz_0}\ .\end{aligned}\tag{13.31}$$

It is easy to see that the determinant of the homogeneous system of equations (13.31) is $(\omega^2-c_1^2K^2)(\omega^2-c_2^2K^2)$, where $K^2=q^2+k^2$. Therefore, the free solution is

$$\begin{aligned}f_0&=iqAe^{ic_1Kt}+kBe^{ic_2Kt}+c.c.\ ,\\ g_0&=kAe^{ic_1Kt}+iqBe^{ic_2Kt}+c.c.\ ,\end{aligned}\tag{13.32}$$

where the constants A and B are determined by the initial conditions. In our case, the force which is proportional to $\delta(t)$ determines the inital conditions, and it is sufficient to limit ourselves to the particular solution of equations (13.31). The particular solution of these equations is

$$f=\tfrac{iq}{\omega^2-c_1^2K^2}\tfrac{mT}{2\pi}e^{-iqz_0}\ ,\ g=\tfrac{k}{\omega^2-c_1^2K^2}\tfrac{mT}{2\pi}e^{-iqz_0}\ .\tag{13.33}$$

The ω-Fourier transform gives

$$\begin{aligned}f&=-\tfrac{iq}{c_1K}\theta(t)\sin c_1Kt\cdot\tfrac{mT}{2\pi}e^{-iqz_0}\ ,\\ g&=-\tfrac{k}{c_1K}\theta(t)\sin c_1Kt\cdot\tfrac{mT}{2\pi}e^{-iqz_0}\ .\end{aligned}\tag{13.34}$$

We leave aside the factor $\theta(t)$ (Heaviside function 1 for $t>0$ and 0 for $t<0$). Further on, it is convenient to do first the k-integration, according to equations (13.17) and (13.18). We limit ourselves to the vertical component $u_z(q;t,r)$, which implies the integral[20]

$$\begin{aligned}&\int_0^{+\infty}dkk\tfrac{1}{K}\sin c_1Kt\cdot J_0(kr)=\\ &=\int_{|q|}^{+\infty}dK\sin c_1Kt\cdot J_0(r\sqrt{K^2-q^2})=\\ &=\begin{cases}0\ ,\ 0<c_1t<r\ ,\\ \frac{\cos\left(|q|\sqrt{c_1^2t^2-r^2}\right)}{\sqrt{c_1^2t^2-r^2}}\ 0<r<c_1t\ .\end{cases}\end{aligned}\tag{13.35}$$

[20] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, 6th ed.,Academic, NY (2000), p. 714, 6.677 (1).

Finally, the vertical displacement is given by

$$u_z(t,r,z) = \frac{mT}{(2\pi)^2 c_1} \frac{\theta(c_1 t - r)}{\sqrt{c_1^2 t^2 - r^2}} \cdot \\ \cdot \int dq(-iq) e^{iq(z-z_0)} \cos\left(q\sqrt{c_1^2 t^2 - r^2}\right) \ . \tag{13.36}$$

This integral implies a derivative $\partial/\partial z$ and the functions $\delta(z - z_0 \pm \sqrt{c_1^2 t^2 - r^2})$; these functions can be transformed in $\delta(t - R/c_1)$, where $R = \sqrt{r^2 + (z - z_0)^2}$. In the far-field zone we get

$$u_z(t,R) = -\frac{mT(z - z_0)}{4\pi c_1 R^2} \delta'(R - c_1 t) \ , \tag{13.37}$$

which is precisely the result obtained previously for seismic waves.[21]

13.2 Vibrations of an elastic sphere

13.2.1 Solid sphere

The long-standing interest in the vibrations of a solid sphere is related to the seismic vibrations of the Earth.[22] After a relatively short burst of energy in an earthquake the Earth continues to vibrate freely for a long time. Though with a liquid outer core and a viscous mantle, the Earth is still approximated by a solid sphere. The great progress in studying the vibrations of a homogeneous and isotropic elastic sphere was made since the beginning, when Lamb introduced the vector spherical harmonics (Hansen vectors).[23] The relevant eingenfrequencies were computed numerically as early as 1898.[24] We discuss herein a natural simplification of this problem, which arises from

[21] See, for instance, B. F. Apostol, *Seismology*, Nova, NY (2020).

[22] K. Aki and P. G. Richards, *Quantitative Seismology*, University Science Books, Sausalito, CA (2009); A. Ben-Menahem and J. D. Singh, Seismic Waves and Sources, Springer, New York (1981).

[23] H. Lamb, "On the vibrations of an elastic sphere", Proc. London Math. Soc. **13** 189 (1882); "On the oscillations of a viscous spheroid", Proc. London Math. Soc. **13** 51 (1881); W. W. Hansen, "A new type of expansion in radiation problems", Phys. Rev. **47** 139 (1935).

[24] T. J. I'A. Bromwich, "On the influence of gravity on elastic waves, and, in particular, on the vibrations of an elastic globe", Proc. London Math. Soc. **30** 98 (1898).

the fact that a large radius of the sphere is a natural cutoff. Apart from giving formally the general solution of vibrations generated by the seismic tensorial force, we show that a large radius simplifies appreciably the boundary conditions, leading readily to the estimation of the eigenfrequencies (normal modes). The particular case of a fluid sphere is treated to a larger extent.

The elastic vibrations of a homogeneous and isotropic solid are described by the equation

$$\mu curl\, curl\mathbf{u} - (\lambda + 2\mu) grad\, div\mathbf{u} - \rho\omega^2 \boldsymbol{u} = \mathbf{F}(\omega) \ , \tag{13.38}$$

where $\boldsymbol{u}$ is the local displacement, ρ is the density, μ and λ are the Lame elastic moduli, ω is the frequency, and $\boldsymbol{F}(\omega)$ is the force.[25] The components of the seismic tensorial force are

$$F_i(\omega) = M_{ij}(\omega)\partial_j \delta(\boldsymbol{r} - \boldsymbol{r}_0) \ , \tag{13.39}$$

where $M_{ij}(\omega)$ is the Fourier transform of the seismic moment, $\boldsymbol{r}_0$ is the position of the point where the force is placed and $i, j, ... = 1, 2, 3$ are cartesian labels.[26] An equivalent form of equation (13.38) is

$$c_2^2 \Delta \boldsymbol{u} + (c_1^2 - c_2^2) grad\, div\boldsymbol{u} + \omega^2 \boldsymbol{u} = -\boldsymbol{f} \ , \tag{13.40}$$

where $c_1 = \sqrt{(\lambda + 2\mu)/\rho}$ is the velocity of the longitudinal elastic waves, $c_2 = \sqrt{\mu/\rho}$ is the velocity of the transverse elastic waves and $\boldsymbol{f}(\omega) = \boldsymbol{F}(\omega)/\rho$ (also, $m_{ij}(\omega) = M_{ij}(\omega)/\rho$). As it is well known, equation (13.40) is separated in two inhomogeneous Helmholtz equations

$$c_1^2 \Delta\Phi + \omega^2 \Phi = -\varphi \ , \ c_2^2 \Delta \boldsymbol{A} + \omega^2 \boldsymbol{A} = -\boldsymbol{h} \ , \tag{13.41}$$

by $\boldsymbol{u} = grad\Phi + curl\boldsymbol{A}$ ($div\boldsymbol{A} = 0$), $\boldsymbol{f} = grad\varphi + curl\boldsymbol{h}$ ($divh = 0$), where φ and $\boldsymbol{h}$ are given by $\Delta\varphi = div\boldsymbol{f}$, $\Delta\boldsymbol{h} = -curl\boldsymbol{f}$ (Helmholtz potentials). We get

$$\varphi = -\frac{1}{4\pi} m_{ij} \partial_i \partial_j \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_0|} \ , \ h_i = \frac{1}{4\pi} \varepsilon_{ijk} m_{kl} \partial_j \partial_l \frac{1}{|\boldsymbol{r} - \boldsymbol{r}_0|} \tag{13.42}$$

[25] L. Landau and E. Lifshitz, *Course of Theoretical Physics*, *Theory of Elasticity*, vol. 7, Elsevier, Oxford (1986).

[26] B. F. Apostol, "Elastic waves inside and on the surface of an elastic half-space", Quart. J. Mech. Appl. Math. **70** 281 (2017); "Elastic displacement in a half-space under the action of a tensor force. General solution for the half-space with point forces", J. Elast. **126** 231 (2017).

(where ε_{ijk} is the totally antisymmetric tensor of rank three), such that we are led to consider the equation

$$c^2\Delta F + \omega^2 F = \frac{1}{r} \tag{13.43}$$

with solution

$$F(r) = \frac{1-\cos kr}{\omega^2 r}\ ,\ k^2 = \omega^2/c^2\ ; \tag{13.44}$$

This solution results immediately from the vibration Green function $G = -\frac{\cos kr}{4\pi c^2 r}$ of the Helmholtz equation $c^2\Delta G + \omega^2 G = \delta(\boldsymbol{r})$. We get a particular solution of equation (13.40)

$$\begin{aligned} u_i^p = \tfrac{1}{4\pi} m_{ij}\partial_j \Delta F_2(\mid \boldsymbol{r}-\boldsymbol{r}_0 \mid)+ \\ +\tfrac{1}{4\pi} m_{jk}\partial_i\partial_j\partial_k \left[F_1(\mid \boldsymbol{r}-\boldsymbol{r}_0 \mid) - F_2(\mid \boldsymbol{r}-\boldsymbol{r}_0 \mid)\right]\ . \end{aligned} \tag{13.45}$$

For a fluid, where $c_2 = 0$ ($\mu = 0$) and $m_{ij} = -m\delta_{ij}$, this solution becomes

$$\boldsymbol{u}^p = -\frac{m}{4\pi c_1^2} grad \frac{\cos k_1 \mid \boldsymbol{r}-\boldsymbol{r}_0 \mid}{\mid \boldsymbol{r}-\boldsymbol{r}_0 \mid}\ . \tag{13.46}$$

In order to apply these results to a sphere we need to use expansions in series of (orthogonal) vector spherical harmonics, defined by[27]

$$\begin{aligned} \mathbf{R}_{lm} &= Y_{lm}\mathbf{e}_r\ , \\ \mathbf{S}_{lm} &= \tfrac{\partial Y_{lm}}{\partial\theta}\mathbf{e}_\theta + \tfrac{1}{\sin\theta}\tfrac{\partial Y_{lm}}{\partial\varphi}\mathbf{e}_\varphi\ , \\ \mathbf{T}_{lm} &= \tfrac{1}{\sin\theta}\tfrac{\partial Y_{lm}}{\partial\varphi}\mathbf{e}_\theta - \tfrac{\partial Y_{lm}}{\partial\theta}\mathbf{e}_\varphi\ , \end{aligned} \tag{13.47}$$

$l \neq 0$, where Y_{lm} are spherical harmonics and $\mathbf{e}_{r,\theta,\varphi}$ are the spherical unit vectors. The functions $\boldsymbol{R}_{lm}$ and $\boldsymbol{S}_{lm}$ are called spheroidal functions, while the functions $\boldsymbol{T}_{lm}$ are called toroidal functions. The series expansion reads

$$\mathbf{u}^p = \sum_{lm}(f_{lm}^p\mathbf{R}_{lm} + g_{lm}^p\mathbf{S}_{lm} + h_{lm}^p\mathbf{T}_{lm})\ , \tag{13.48}$$

[27] R. G. Barrera, G. A. Estevez and J. Giraldo, "Vector spherical harmonics and their applications to magnetostatics", Eur. J. Phys. **6** 287 (1985).

where f^p_{lm}, g^p_{lm} and h^p_{lm} are functions only of the radius r. A similar series holds also for the free solution $\boldsymbol{u}^f$ of equation (13.38).

The explicit form of the coefficients f^p_{lm}, g^p_{lm} and h^p_{lm} is extremely cumbersome. We prefer to work formally with equation (13.38) and series expansions of the full solution $\boldsymbol{u} = \boldsymbol{u}^p + \boldsymbol{u}^f$ and the force $\boldsymbol{F}(\omega)$, with coefficients f_{lm}, g_{lm} and h_{lm} and $F^{r,s,t}$, respectively. Making use of such series expansions and the properties of the vector spherical harmonics, we get the equations

$$\begin{gathered}
f^{''} + \tfrac{2}{r}f^{'} + \tfrac{\rho\omega^2}{\lambda+2\mu}f - \left[2 + \tfrac{\mu l(l+1)}{\lambda+2\mu}\right]\tfrac{1}{r^2}f + \\
+\tfrac{(\lambda+3\mu)l(l+1)}{(\lambda+2\mu)r^2}g - \tfrac{(\lambda+\mu)l(l+1)}{(\lambda+2\mu)r}g^{'} = -\tfrac{F^r}{\lambda+2\mu} \ , \\
g^{''} + \tfrac{2}{r}g^{'} + \tfrac{\rho\omega^2}{\mu}g - \tfrac{(\lambda+2\mu)l(l+1)}{\mu r^2}g + \\
+\tfrac{2(\lambda+2\mu)}{\mu r^2}f + \tfrac{\lambda+\mu}{\mu r}f^{'} = -\tfrac{F^s}{\mu} \ , \\
h^{''} + \tfrac{2}{r}h^{'} + \tfrac{\rho\omega^2}{\mu}h - \tfrac{l(l+1)}{r^2}h = -\tfrac{F^t}{\mu} \ ,
\end{gathered} \tag{13.49}$$

where, for the sake of simplicity, we dropped out the suffixes lm.

We turn now to the boundary conditions. The force $\mathbf{P}$ acting (inwards) on the surface $r = R$ of the sphere, where R is the radius of the sphere, with the spherical components P_α ($\alpha = r, \theta, \varphi$) is $P_\alpha = n_\beta \sigma_{\alpha\beta} = \sigma_{\alpha r}$, where the stress tensor is given by $\sigma_{\alpha\beta} = 2\mu u_{\alpha\beta} + \lambda u_{\gamma\gamma}\delta_{\alpha\beta}$; we get

$$\begin{gathered}
2\mu u_{\theta r} = P_\theta \ , \ 2\mu u_{\varphi r} = P_\varphi \ , \\
2\mu u_{rr} + \lambda div\mathbf{u} = P_r
\end{gathered} \tag{13.50}$$

(for $r = R$), where $div\mathbf{u}$ is written in spherical coordinates,

$$div\mathbf{u} = \sum_{lm}\left[\frac{1}{r^2}\frac{d}{dr}(r^2 f_{lm}) - \frac{g_{lm}}{r}l(l+1)\right]Y_{lm} \tag{13.51}$$

(by using the properties of the vector spherical harmonics equations).

We compute the strain tensor $u_{\alpha\beta}$ in spherical coordinates[28]

$$u_{rr} = \frac{\partial u_r}{\partial r} \ , \ u_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \ ,$$

$$u_{\varphi\varphi} = \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial \varphi} + \frac{u_\theta}{r}\cot\theta + \frac{u_r}{r} \ ,$$

$$2u_{\theta\varphi} = \frac{1}{r}\left(\frac{\partial u_\varphi}{\partial \theta} - u_\varphi \cot\theta\right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \varphi} \ , \tag{13.52}$$

$$2u_{r\theta} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} \ ,$$

$$2u_{\varphi r} = \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \varphi} + \frac{\partial u_\varphi}{\partial r} - \frac{u_\varphi}{r}$$

by using the spherical components

$$u_r = \sum_{lm} f_{lm} Y_{lm} \ ,$$

$$u_\theta = \sum_{lm} g_{lm}\frac{\partial Y_{lm}}{\partial \theta} + \sum_{lm}\frac{h_{lm}}{\sin\theta}\frac{\partial Y_{lm}}{\partial \varphi} \ , \tag{13.53}$$

$$u_\varphi = \sum_{lm}\frac{g_{lm}}{\sin\theta}\frac{\partial Y_{lm}}{\partial \varphi} - \sum_{lm} h_{lm}\frac{\partial Y_{lm}}{\partial \theta}$$

of the expansion of the displacement vector and the definition of the vector spherical functions (equations (13.47)). Similarly, we decompose the force $\mathbf{P}$ in vector spherical harmonics (with coefficients $P^{r,s,t}$) and identify its spherical components. The boundary conditions given by equations (13.50) lead to

$$2\mu f' + \lambda\left[\frac{2}{r}f + f' - \frac{g}{r}l(l+1)\right] |_{r=R} = P^r \ ,$$

$$\mu\left(\frac{g}{r} - g' - \frac{f}{r}\right) |_{r=R} = -P^s \ , \tag{13.54}$$

$$\mu\left(\frac{h}{r} - h'\right) |_{r=R} = -P^t \ ,$$

where we dropped the subscripts lm.

[28] L. Landau and E. Lifshitz, *Course of Theoretical Physics*, *Theory of Elasticity*, vol. 7, Elsevier, Oxford (1986).

13.2.2 Vibration eigenfrequencies for large radius

The solutions f, g and h of equations (13.49) consist of free solutions (solutions of the homogeneous equations (13.49)) plus particular solutions. The homogeneous third equation (13.49), which describes toroidal vibrations, is the equation of the spherical Bessel functions $j_l(kr)$, $k = \sqrt{\rho\omega^2/\mu} = \omega/c_2$. For $F^t = 0$ and $P^t = 0$ (a free surface) the third equation in the boundary conditions (13.54) gives

$$j_l(kR) = kRj_l^{'}(kR) \; ; \tag{13.55}$$

this equation has an infinity of solutions β_{ln}, labelled by integer n, such that we get the eigenfrequencies

$$\omega_{ln} = \frac{c_2}{R}\beta_{ln} \; . \tag{13.56}$$

We can get an estimate of the numbers β_{ln} by using the asymptotic expression of the spherical Bessel functions[29]

$$j_l(kr) \simeq \frac{1}{kr}\cos\left[kr - (l+1)\frac{\pi}{2}\right] \; , \; kr \gg 1 \; ; \tag{13.57}$$

for $kR \gg 1$ equation (13.55) becomes

$$\tan\left[kR - (l+1)\frac{\pi}{2}\right] = -\frac{2}{kR} \; , \tag{13.58}$$

which has the approximate zeros

$$\beta_{ln} \simeq n\pi + (l+1)\frac{\pi}{2} \; , \tag{13.59}$$

where n is any (large) integer. We can see that the frequencies are dense for large R ($\Delta\omega_{ln} = \pi c_2/R$). The free toroidal solution is a superposition of $j_l(k_{ln}r)$, where $k_{ln} = \omega_{ln}/c_2 = \beta_{ln}/R$, with undetermined coefficients.

In general, for $F^t \neq 0$ and $P^t \neq 0$ the free toroidal solution is $C_l j_l(kr)$, where the constants C_l are determined from the boundary condition.

[29] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, National Bureau of Standards, USA Government Printing Office, Washington (1964).

It is easy to see that these coefficients include singular factors proportional to $\sim \frac{1}{\omega-\omega_{ln}}$, such that, the integration over frequencies leads to toroidal vibrations governed by the eigenfrequencies ω_{ln}. In general, the solution h_{lm} depends on two integration constants, which are determined by the boundary condition and the condition of a finite solution at the origin.

We pass now to the spheroidal components which involve the functions f and g in equations (13.49) and (13.54). We note that the two coupled equations (13.49) for the functions f and g include Bessel operators for spherical Bessel functions. We can get a simplified picture of these equations for large values of r. Indeed, it is easy to see that in the limit $\omega r/c_{1,2} \gg l^2$ the free solutions are

$$f \simeq \frac{A}{r}\cos(\omega r/c_1 + \varphi_{1l}) \ , \ g \simeq \frac{B}{r}\cos(\omega r/c_2 + \varphi_{2l}) \ , \qquad (13.60)$$

where the coefficients A and B and the phases $\varphi_{1l,2l}$ remain undetermined. In this limit the boundary conditions are $f' \mid_{r=R} = g' \mid_{r=R} = 0$ and the eigenfrequencies are given by $\omega_{nl}R/c_{1,2} + \varphi_{1l,2l} = n\pi$, where n is any integer (the roots of the equation $\sin(\omega R/c_{1,2} + \varphi_{1l,2l}) = 0$). The condition $\omega r/c_{1,2} \gg l^2$ is satisfied for a large r and a reasonably large range of frequencies and parameter l. For instance, for r close to Earth's surface, which is the spatial region of interest, we get $\omega R/c$ of the order $\simeq 10^3\omega$ for a mean radius of the Earth $R = 6370km$ and a mean velocity of the elastic waves $c = 5km/s$. Indeed, frequencies as low as $\omega = 10^{-3}s^{-1}$ are known for Earth's seismic vibrations.[30]

We can see that there are two branches of spheroidal eigenfrequencies (corresponding to the velocities $c_{1,2}$), which are dense (continuous) for large R, very similar with the infinite space (as expected for large R); the $\omega^{(2)}$-branch, although close to the toroidal branch, is distinct (there is a total of three branches of eigenfrequencies, corresponding to the three degrees of freedom; in the limit of the rotations of the

[30] J. Mendiguren, "Identification of free oscillation spectral peaks for 1970 July 31 Colombian deep shock using the excitation criterion", Geophys. J. Roy. Astron. Soc. **33** 281 (1973); "High resolution spectroscopy of the Earth's free oscillations, knowing the earthquake source mechanism", Science **179** 179 (1973); F. Gilbert and A. M. Dziewonski, "An application of the normal mode theory to the retrieval of the structural parameters and source mechanisms from seismic spectra", Phil. Trans. Roy. Soc. (London) **A278** 187 (1975).

sphere as a whole their frequencies go to zero (acoustic modes)). For non-vanishing forces we have spheroidal vibrations driven by these forces, as discussed for the previous cases. The set of all eigenfrequencies is called the (seismic) spectrum. Earth's eigenmodes with eigenfrequencies of the order $10^{-3} - 10^{-4}s^{-1}$, excited by earthquakes, are known.

The numerical solution of equations (13.49) indicates that the lowest mode (the fundamental mode) is $\mathbf{S}_{lm}$ with $l = 2$ and $n = 0$ (therefore, we may denote it as $S^{(n=0)}_{l=2,m}$);[31] it is denoted by ${}_0S_2$, and its eigenfrequency is denoted ω_{20}; the corresponding period is approximately 1 hour. Much later, the Earth's crust was modelled as a series of superposed layers, with welded interfaces; the vibrations of such a stack of layers can be computed and long periods of the fundamental modes have been obtained; the dispersion relation of these modes (*i.e.*, the dependence of the frequency on their label n) can give information about the inner crustal structure.[32] The first observation of "free oscillations of the Earth as a whole" was made for the Kamchatka earthquake of November 4, 1952;[33] they were followed by many observations of the Earth's vibrations caused by the great Chile earthquake of May 22, 1960[34] (with magnitude greater than 8, which saturated the scales[35]). Today, eigenoscillations of the Earth can be recorded even for small earthquakes.[36]

[31] T. J. I'A. Bromwich, "On the influence of gravity on elastic waves, and, in particular, on the vibrations of an elastic globe", Proc. London Math. Soc. **30** 98 (1898).

[32] N. A. Haskell, "The dispersion of surface waves in multilayered media", Bul. Seism. Soc. Am. **43** 17 (1953); M. Ewing, W. Jardetzky and F. Press, *Elastic Waves in Layered Media,* McGraw-Hill, NY (1957).

[33] H. Benioff, B. Gutenberg and C. F. Richter, *Progr. Report*, Trans. Am. Geophys. Union **35** 979 (1954).

[34] H. Benioff, F. Press and S. W. Smith, "Excitation of the free oscillations of the Earth by earthquakes", J. Geophys. Res. **66** 605 (1961); N. F. Ness, J. C. Harrison and L. B. Slichter, "Observations of the free oscillations of the Earth", J. Geophys. Res. **66** 621 (1961); L. E. Alsop, G. H. Sutton and M. Ewing, "Free oscillations of the Earth observed on strain and pendulum seismographs", J. Geophys. Res. **66** 631 (1961).

[35] H. Kanamori, "The energy release in great earthquakes", J. Geophys. Res. **82** 2981 (1977).

[36] B. Block, J. Dratler and R. D. Moore, "Earth and normal modes from a 6.5 magnitude earthquake", Nature **226** 343 (1970).

From studies of propagation of the seismic waves it was inferred the Earth's solid inner core[37] of radius $\simeq 1000km$ and the outer liquid core of radius $\simeq 2000km$. The inner-outer core discontinuity is called the Bullen, or Lehmann, discontinuity. The temperature of the inner core is $\simeq 6000K$ (iron and nickel) and the pressure is $\simeq 10^{12}dyn/cm^2$. The buoyancy at this boundary could be the source of convection currents which generate the Earth's magnetic field (geodynamo effect). The next layers are a viscous mantle of thickness $\simeq 3000km$ and the solid crust of thickness $\simeq 70km$. The boundary between mantle and crust is known as the Mohorovicic discontinuity.

13.2.3 Fluid Sphere

For a fluid sphere the shear modulus μ is zero ($\mu = 0$); equations (13.49) become

$$\begin{gathered} f^{''} + \tfrac{2}{r}f^{'} + k^2 f - \tfrac{2}{r^2}f - \tfrac{d}{dr}\left[\tfrac{l(l+1)g}{r}\right] = -\tfrac{F^r}{\lambda} \\ \tfrac{1}{r}f^{'} + \tfrac{2}{r^2}f - \tfrac{l(l+1)}{r^2}g + k^2 g = -\tfrac{F^s}{\lambda} \ , \end{gathered} \tag{13.61}$$

where $k^2 = \rho\omega^2/\lambda = \omega^2/c^2$; the boundary condition reads

$$\left[\frac{2}{r}f + f^{'} - \frac{g}{r}l(l+1)\right] |_{r=R} = \frac{P^r}{\lambda} \ . \tag{13.62}$$

Let us introduce $div\mathbf{u}$, given by equation (13.51), which includes

$$d = f^{'} + \frac{2}{r}f - \frac{g}{r}l(l+1) \ . \tag{13.63}$$

Then the boundary condition becomes

$$d\ |_{r=R} = \frac{P^r}{\lambda} \ , \tag{13.64}$$

[37] I. Lehmann, "$P^{'}$". Publications du Bureau Central Seismologique International, Serie A, Travaux Scientifiques, **14** 87 (1936); A. M. Dziewonski and F. Gilbert, "Solidity of the inner core of the Earth inferred from normal mode observations", Nature **234** 465 (1971).

the second equation (13.61) reads

$$\frac{d}{r} + k^2 g = -\frac{F^s}{\lambda} \tag{13.65}$$

and the first equation (13.61) is

$$d^{'} + k^2 f = -\frac{F^r}{\lambda} \ . \tag{13.66}$$

Hence, we have

$$g = -\frac{d}{k^2 r} - \frac{F^s}{\lambda k^2} \ , \ f = -\frac{d^{'}}{k^2} - \frac{F^r}{\lambda k^2} \ . \tag{13.67}$$

Now we introduce these functions in equation (13.63) and get

$$d^{''} + \frac{2d^{'}}{r} + k^2 d - \frac{l(l+1)}{r^2} d = -\frac{(F^r)^{'}}{\lambda} - \frac{2F^r}{\lambda r} + \frac{F^s}{\lambda r} l(l+1) \ . \tag{13.68}$$

For free vibrations this is the Bessel equation for spherical Bessel functions $d = j_l(kr)$; the boundary condition (13.64) leads to the eigenfrequencies $\omega_{ln} = (c/R)\beta_{ln}$, $j_l(\beta_{ln}) = 0$. In a fluid we have only pressure p, and the stress tensor is $\sigma_{ij} = -p\delta_{ij}$ ($\sigma_{ij} = 2\mu u_{ij} + \lambda u_{kk}\delta_{ij}$ with $\mu = 0$); therefore, for a fluid $p = -\lambda u_{ii} = -\lambda div\mathbf{u}$; equations written above for d are in fact equations for the pressure p. It is convenient to introduce the decomposition in Helmholtz potentials $\mathbf{u} = grad\Phi + curl\mathbf{A}$, $div\mathbf{A} = 0$ and $\mathbf{F} = grad\varphi + curl\mathbf{h}$, $div\boldsymbol{h} = 0$; then, $p = -\lambda\Delta\Phi$ and the equation of motion $\rho\ddot{\mathbf{u}} = \lambda grad \cdot div\mathbf{u} + \mathbf{F} = -gradp + \mathbf{F}$ becomes $\rho\ddot{\Phi} = \lambda\Delta\Phi + \varphi$, where the potential φ is given by $\Delta\varphi = div\boldsymbol{F}$ and $\boldsymbol{h} = 0$, $\boldsymbol{A} = 0$. For vibrations this equation reads $c^2\Delta\Phi + \omega^2\Phi = -\frac{1}{\rho}\varphi$ and for $\boldsymbol{F} = -Mgrad\delta(\boldsymbol{r} - \boldsymbol{r}_0)$ we get $\varphi = -M\delta(\boldsymbol{r} - \boldsymbol{r}_0)$, $\Phi = -m\frac{\cos k|\boldsymbol{r}-\boldsymbol{r}_0|}{4\pi c^2|\boldsymbol{r}-\boldsymbol{r}_0|}$ ($m = M/\rho$) and the solution $\boldsymbol{u}^p$ given by equation (13.46).

13.2.4 Static self-gravitation

A gravitational force

$$FdV = G\rho dV\frac{m}{r^2} = \frac{4\pi}{3}G\rho^2 rdV \tag{13.69}$$

acts upon a volume element dV placed at distance r from the centre of a sphere, where $G = 6.67 \times 10^{-8} cm^3/g \cdot s^2$ is the universal constant of gravitation, ρ is the density of the sphere (assumed incompressible) and $m = (4\pi/3)\rho r^3$ is the mass of the sphere with radius r. If the sphere is compressible, the gravitational potential φ is given by the Poisson equation $\Delta\varphi = 4\pi G\rho$ and the gravitational force per unit mas is $\mathbf{F} = -grad\varphi$; the condition of (hydrostatic) equilibrium (for a non-rotationg sphere) reads $gradp = \rho\mathbf{F} = -\rho grad\varphi$, such that $div\left[(gradp)/\rho\right] = -4\pi G\rho$; the dependence of the pressure on the density is given by the equation of state; for a constant density the pressure for a self-gravitating sphere of radius R at rest with free surface is $p = (2\pi/3)G\rho^2(R^2 - r^2)$ (it seems that the pressure in the inner Earth's (solid) core is $\simeq 300GPa = 3 \times 10^{12} dyn/cm^2$). Making use of equation (13.69), the equation of the elastic motion reads

$$\rho\ddot{\mathbf{u}} - \mu\Delta\mathbf{u} - (\lambda + \mu)grad\, div\mathbf{u} = \mathbf{F} = -\gamma\mathbf{r} \ , \tag{13.70}$$

where $\gamma = (4\pi/3)G\rho^2$. Since $Y_{00} = 1/\sqrt{4\pi}$, we may write

$$\mathbf{F} = -\gamma\mathbf{r} = -\sqrt{4\pi}\gamma r Y_{00}\mathbf{e}_r \ , \tag{13.71}$$

whence we can see that $\mathbf{F}$ has a series expansion of spheroidal and toroidal functions with all the coefficients zero, except the coefficient $F_{00}^r = -\sqrt{4\pi}\gamma r$ of the function $\mathbf{R}_{00}$; it follows that the motion may include all the eigenmodes $\mathbf{S}_{lm}$ and $\mathbf{T}_{lm}$, as well as all the eigenmodes $\mathbf{R}_{lm}$, the latter with $l \neq 0$; for $l = 0$, $m = 0$ the motion, described by $f = f_{00}$, is driven by the gravitational force. We note also that the force in equation (13.70) is static, which means that its Fourier transform is proportional to $\delta(\omega)$. For $l = 0$ the first equation (13.49) includes only the function f, *i.e.* $f\delta(\omega)$; this equation reads

$$f^{''} + \tfrac{2}{r}f^{'} - \tfrac{2}{r^2}f = \tfrac{\sqrt{4\pi}\gamma}{\lambda+2\mu}r \ . \tag{13.72}$$

It is easy to see that a particular solution of this equation is $[\sqrt{4\pi}\gamma/10(2\mu + \lambda)]r^3$, while the homogeneous part of this equation has the solution $C_1 r + C_2/r^2$, where $C_{1,2}$ are constants of integration; we must take $C_2 = 0$, because the solution is finite at the origin. We are left with the solution

$$u_r = Ar^3 + C_1 r \ , \ A = \frac{\gamma}{10(2\mu + \lambda)} \ . \tag{13.73}$$

This solution must satisfy the boundary conditions at the surface of the sphere; making use of equations (13.53), we have the strain tensor $u_{rr} = u_r^{'}$ and $u_{\theta\theta} = u_{\varphi\varphi} = u_r/r$; the force on the surface is $-\sigma_{\alpha r} \mid_R$, where the stress tensor is given by $\sigma_{\alpha\beta} = 2\mu u_{\alpha\beta} + \lambda u_{\gamma\gamma\delta_{\alpha\beta}}$; for a free surface we get the boundary condition

$$(2\mu + \lambda)u_r^{'} + 2\lambda\frac{u_r}{r} \mid_{r=R}= 0 \tag{13.74}$$

($\sigma_{\alpha r} \mid_R= 0$), whence we determine the constant $C_1 = -[(6\mu+5\lambda)/(2\mu+3\lambda)]AR^2$ and, finally, the radial displacement

$$\begin{aligned} u_r &= Ar\left(r^2 - \tfrac{6\mu+5\lambda}{2\mu+3\lambda}R^2\right) = \\ &= \tfrac{\gamma}{10(2\mu+\lambda)}r\left(r^2 - \tfrac{6\mu+5\lambda}{2\mu+3\lambda}R^2\right) \ ; \end{aligned} \tag{13.75}$$

we note that the radial displacement u_r is negative, as expected. It is worth estimating the radial displacement at the surface due to gravitation

$$u_r \mid_{r=R}= -\frac{\gamma}{5(2\mu + 3\lambda)}R^3 \ ; \tag{13.76}$$

making use of $\rho = 5g/cm^3$, λ, $\mu \simeq 10^{11}dyn/cm^2$ (parameters for Earth), we get $\gamma \simeq 10^{-6}g/cm^3s^2$and $u_r \mid_R\simeq 10^{-18}R^3cm \simeq 10^8cm = 10^3km$, for the Earth's radius $R \simeq 6 \times 10^8cm$; this is a distance of the order of the Earth radius. Moreover, the strain is of the order 1/6, which may cast doubts on the validity of the linear elasticity used in this estimation. In addition, we note that the density suffers an important change due to the static gravitational field. Indeed, the change in density is $\delta\rho = -div(\rho\mathbf{u}) = -\rho_0 div\mathbf{u}$, where ρ_0 is the uniform initial density; with $\mathbf{u}$ given by equation (13.75) we get

$$\frac{\delta\rho}{\rho_0} = A(3\alpha R^2 - 5r^2) \ , \ \alpha = \frac{6\mu + 5\lambda}{2\mu + 3\lambda} \ , \tag{13.77}$$

which is of the order unity. The proper estimation of the static effect of the self-gravitational field on the elastic sphere is to solve simultaneously the equation of elastic equilibrium (13.70) with $\mathbf{F} = -\rho grad\varphi$ and the Poisson equation for the gravitational field φ, $\Delta\varphi = 4\pi G\rho$. With spherical symmetry we have

$$\mathbf{F} = -\frac{4\pi}{3r^2}G\rho\int_{0<r'<r} d\mathbf{r}^{'}\rho\frac{\mathbf{r}}{r} \ ; \tag{13.78}$$

the Poisson equation for the gravitational potential may be written as $\Delta(\mathbf{F}/\rho) = -4\pi G grad\rho$, such that the problem involves two equations and unknowns, $\mathbf{u}$ and ρ. Since this is a more difficult problem it is preferable to consider the density ρ as an empirically known function of r (a parametrization in powers of r can be used for ρ and a variational approach can be applied to the problem). Even so, the equations governing the influence of the gravitational field upon the elasticity of a self-gravitating sphere are difficult.

13.2.5 Dynamic self-gravitation

Let us assume a spheric, non-rotating, homogeneous, elastic Earth at equilibrium under the action of its own gravitational field; we consider small elastic deformations of this equilibrium state; in first approximation, we have a small change denoted by K in the gravitational potential as a consequence of the small changes in density $-div(\rho\mathbf{u})$, *i.e.,* we have

$$\Delta K = -4\pi G div(\rho\mathbf{u}) \ , \tag{13.79}$$

where ρ is a known function of r. The equation of elastic motion reads

$$\rho\ddot{\mathbf{u}} - \mu\Delta\mathbf{u} - (\lambda+\mu) grad\, div\mathbf{u} = -\rho grad K \ . \tag{13.80}$$

These two coupled (vectorial) equations are difficult to be treated by an analytical method, due to the non-uniformity of the density. For a uniform density, taking the *div* in equation (13.80) and using equation (13.79) we get for $D = div\mathbf{u}$

$$\rho\ddot{D} - (\lambda+2\mu)\Delta D = 4\pi G\rho^2 D \ , \tag{13.81}$$

an equation which indicates that the frequency ω changes by

$$\Delta(\omega^2) = -4\pi G\rho \ ; \tag{13.82}$$

for frequencies as low as $\omega = 10^{-4}s^{-1}$ the variation given by equation (13.82) is large. Let us use the Helmholtz decomposition $\mathbf{u} = grad\Phi + curl\mathbf{A}$, $div\mathbf{A} = 0$; then, from equation (13.79) we have $K = -4\pi G\rho\Phi$ and from equation (13.80) we get $\Delta\Phi + k_1^2\Phi = 0$, $\Delta\mathbf{A} + k_2^2\mathbf{A} = 0$. These are the same equations as those which hold in the absence of the gravitational field, except that k_1^2 is changed into $k_1^2 \rightarrow k_1^2 + 4\pi\rho G/c_1^2$. Moreover, we can see that only the spheroidal modes are affected by gravitation (since $div\mathbf{T}_{lm} = 0$). It follows that the spheroidal frequencies (*i.e.*, the branches $\omega^{(1,2)}$) are given by the same relations of the type $\omega = (c/R)\beta$, where β denote the zeros the spherical Bessel functions in the limit of large R; for $c = c_1$, this relation reads $\omega^2 + 4\pi\rho G = (c_1^2/R^2)\beta^2$. Hence, we may see that we should have the inequality $(c_1^2/R^2)\beta^2 > 4\pi\rho G$, or $(\lambda + 2\mu)\beta^2 > 4\pi\rho^2 GR^2$. The term in the right side of this inequality is, up to an immaterial numerical factor, the pressure due to the gravitation at the origin; it is much larger than the elastic pressure $\lambda + 2\mu$. The inequality is not satisfied for small values of β (as required by experimental observations). It follows that the model of an elastic solid Earth is not valid for the interior of the Earth. In those central regions the elasticity is not able to sustain the gravitational pressure. Likely, an additional pressure exists there, which compensates the gravitational pressure. The large dimensions of the mantle and liquid outer core complicates the matter, and such an Earth's model may exhibit very low frequencies (undertones).[38] If so, we may leave aside the effects of the gravitation in estimating the elastic vibrations of the Earth. In this case, with $c = 5km/s$ we get a period $T \simeq (2.2/\beta)$ hours; the smallest zero of j_2' (corresponding approximately to the mode ${}_0S_2$) is $\beta = 3.6$;[39] we get $T \simeq 37$ minutes (for a velocity $c = 3km/s$ the period is $T = 61$ minutes, which agrees with the experimental observations).

13.2.6 Rotation effect

If a vector $\mathbf{a}$ rotates, its change is $\delta\mathbf{a} + \delta\boldsymbol{\alpha} \times \mathbf{a}$, where $\delta\boldsymbol{\alpha}$ is the infinitesimal rotation angle; therefore, its velocity is $\dot{\mathbf{a}} + \boldsymbol{\Omega} \times \mathbf{a}$, where $\boldsymbol{\Omega}$ is the angular velocity; its acceleration is $\ddot{\mathbf{a}} + \dot{\boldsymbol{\Omega}} \times \mathbf{a} + 2\boldsymbol{\Omega} \times \dot{\mathbf{a}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{a})$. Let us apply this relation to the displaced position $\mathbf{a} = \mathbf{r} + \mathbf{u}$; we get the acceleration $\ddot{\mathbf{u}} + \dot{\boldsymbol{\Omega}} \times (\mathbf{r} + \mathbf{u}) + 2\boldsymbol{\Omega} \times \dot{\mathbf{u}} + \boldsymbol{\Omega} \times [\boldsymbol{\Omega} \times (\boldsymbol{r} + \boldsymbol{u})]$; we can see that additional forces appear in rotation: $-2\boldsymbol{\Omega} \times \dot{\mathbf{u}}$ is

[38] C. L. Pekeris and Y. Accad, "Dynamics of the liquid core of the Earth", Phil. Trans. Roy. Soc. London **A273** 237 (1972).

[39] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, National Bureau of Standards, USA Government Printing Office, Washington (1964).

the Coriolis acceleration and $-\mathbf{\Omega}\times[\mathbf{\Omega}\times(\mathbf{r}+\mathbf{u})]$ is the centrifugal acceleration. The Earth rotates with a constant angular velocity $\Omega = 2\pi/T$, $T = 24$ hours, oriented along the z-axis. We write the equation of elastic motion as

$$\rho\ddot{\mathbf{u}} + 2\rho\mathbf{\Omega}\times\dot{\mathbf{u}} = \mathbf{F} \ , \tag{13.83}$$

where $\mathbf{F}$ includes the elastic force (*i.e.*, $F_i = \partial_j\sigma_{ij}$) and other external forces and the centrifugal force is omitted since Ω is much smaller than the eigenfrequencies of the Earth (an estimation of the longest periods of the Earth's eigenmodes gives an order of magnitude $2\pi R/c\beta \simeq 37$ minutes, for the wave velocity $c = 5km/s$ and $\beta = 3.6$ where R is the Earth's radius).

In the absence of the Coriolis force in equation (13.83) we decompose the force $\mathbf{F}$ and the displacement $\mathbf{u}$ in normal modes by using the spheroidal and toroidal functions. Let us focus on one normal mode, for instance a toroidal mode $\mathbf{u}_{lm}^{(n)} = h_l^{(n)}\mathbf{T}_{lm}$, corresponding to the eigenfrequency $\omega_{ln} = (c_2/R)\beta_{ln}$, where β_{ln} is, approximately, a zero of the function $j_l(k^{(n)}R)$; the eigenfunctions $h_l^{(n)}$ are given by the spherical Bessel functions $j_l(k^{(n)}r)$; it is preferrable to multiply these functions by constants and fix these constants such as

$$\int dr\cdot r^2 h_l^{(n)}(r)h_l^{(n')}(r) = \delta_{nn'} \ ; \tag{13.84}$$

we recall that the toroidal functions are orthogonal, *i.e.*

$$\int do\mathbf{T}_{lm}\mathbf{T}_{l'm'}^* = \delta_{ll'}\delta_{mm'} \ . \tag{13.85}$$

Since $\Omega/\omega_{ln} \ll 1$ we solve equation (13.83) by a perturbation-theory method. First, we drop the labels l, m and n and use the notations $\mathbf{u}_{lm}^{(n)} = \mathbf{u}_0$, $\omega_{lm} = \omega_0$; we seek the solution as a series in powers of Ω/ω_0

$$\mathbf{u} = \mathbf{u}_0 + \frac{\Omega}{\omega_0}\mathbf{u}_1 + ... \ , \tag{13.86}$$

where $\mathbf{u}_1$, to be determined, is assumed orthogonal on $\mathbf{u}_0$,[40] with respect to the scalar product defined as the integration over the whole

[40] F. A. Dahlen and M. L. Smith, "The influence of rotation on the free oscillations of the Earth", Phil. Tras. Roy. Soc. London **A279** 583 (1975).

space, *i.e.*

$$\int d\mathbf{r}\mathbf{u}_1\mathbf{u}_0 = 0 \ . \tag{13.87}$$

A similar series is valid for the frequency

$$\omega = \omega_0 + \frac{\Omega}{\omega_0}\omega_1 + \ldots \ . \tag{13.88}$$

Introducing these series in equation (13.83), with time Fourier transforms, we get

$$\begin{gathered} -\rho\omega_0^2\mathbf{u}_0 = \mathbf{F} \ , \\ -\rho\omega_0\Omega\mathbf{u}_1 - 2\rho\Omega\omega_1\mathbf{u}_0 - 2i\rho\omega_0\boldsymbol{\Omega}\times\mathbf{u}_0 = 0 \ ; \end{gathered} \tag{13.89}$$

the first equation (13.89) defines the function $\mathbf{u}_0$; in the second equation (13.89) we take the scalar product with $\mathbf{u}_0$ and use the orthogonality of $\mathbf{u}_0$ with $\mathbf{u}_1$; we get

$$\omega_1 = -\frac{i\omega_0}{l(l+1)}\int d\mathbf{r}\mathbf{e}_z(\mathbf{u}_0\times\mathbf{u}_0^*) \ , \tag{13.90}$$

where we put $\boldsymbol{\Omega} = \Omega\mathbf{e}_z$, $\mathbf{e}_z$ being the unit vector along the z-axis. Here we use $\mathbf{e_z} = \cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta$, $\mathbf{u}_0 = h_l^{(n)}\mathbf{T}_{lm}$ and $\mathbf{T}_{lm}$ from equations (13.47); we get immediately

$$\omega_1 = \omega_0\frac{m}{l(l+1)} \ , \tag{13.91}$$

where m denotes all integers from $-l$ to l. It follows that the frequencies ω_{ln}, which are degenerate with respect to m, are split into $2l+1$ branches

$$\omega_{ln} \to \omega_{ln} + \Omega\frac{m}{l(l+1)} \ ; \tag{13.92}$$

using ω_1 thus determined, we can get $\mathbf{u}_1$ from the second equation (13.89). Higher-order contributions can be obtained in a similar manner. An m-band occurs for each ω_{ln}, of width $2\Omega/(l+1)$, with the separation frequency $\Omega/l(l+1)$. For a typical eigenperiod of 60 minutes the ratio Ω/ω_0 is approximately $1/20 \ll 1$.

13.2.7 Centrifugal force

The equation of the elastic motion for a body in rotation with a (constant) angular velocity $\mathbf{\Omega}$ reads

$$\begin{gathered}\rho\ddot{\mathbf{u}} + 2\rho\mathbf{\Omega}\times\dot{\mathbf{u}} + \rho\mathbf{\Omega}\times[\mathbf{\Omega}\times(\boldsymbol{r}+\mathbf{u})] = \\ = \mu\Delta\mathbf{u} + (\lambda+\mu)grad\,div\mathbf{u} + \mathbf{F}\ ,\end{gathered} \tag{13.93}$$

where $\mathbf{F}$ is an external force. We note that the centrifugal term $\rho\mathbf{\Omega}\times(\mathbf{\Omega}\times\boldsymbol{r})$ is static, so we can write it as

$$\mathbf{F}_c = \rho\mathbf{\Omega}(\mathbf{\Omega}\boldsymbol{r}) - \rho\Omega^2\mathbf{r}\ , \tag{13.94}$$

where we denoted by $\mathbf{F}_c$ the centrifugal force and dropped any other external force ($\mathbf{F}=0$); we may neglect $\mathbf{u}$ in the centrifugal force, since it is very small in comparison to $\mathbf{r}$. The angular velocity is oriented along the z-axis, $\mathbf{\Omega} = \Omega\mathbf{e}_z$. Making use of $\mathbf{e}_z = \cos\theta\mathbf{e}_r - \sin\theta\mathbf{e}_\theta$ and the spherical harmonics

$$Y_{00} = \frac{1}{\sqrt{4\pi}}\ ,\ Y_{20} = \sqrt{\frac{5}{16\pi}}(1-3\cos^2\theta)\ , \tag{13.95}$$

it is easy to see that we can write F_c as a series expansion

$$\mathbf{F}_c = -\rho\Omega^2 r\left(\alpha\mathbf{R}_{00} + 2\beta\mathbf{R}_{20} - \beta\mathbf{S}_{20}\right) \tag{13.96}$$

in spheroidal functions, where $\alpha = 2\sqrt{4\pi}/3$ and $\beta = \sqrt{16\pi/5}$. We seek a similar expansion for the displacement $\mathbf{u}$,

$$\mathbf{u} = f_1\mathbf{R}_{00} + f_2\mathbf{R}_{20} + g\mathbf{S}_{20}\ ; \tag{13.97}$$

equations (13.49) lead to

$$\begin{gathered}f_1^{''} + \tfrac{2}{r}f^{'} - \tfrac{2}{r^2}f_1 = -\tfrac{\rho\Omega^2\alpha}{\lambda+2\mu}r\ , \\ f_2^{''} + \tfrac{2}{r}f_2^{'} - \tfrac{2(\lambda+5\mu)}{\lambda+2\mu}\tfrac{1}{r^2}f_2 + \tfrac{6(\lambda+3\mu)}{\lambda+2\mu}\tfrac{1}{r^2}g - \\ -\tfrac{6(\lambda+\mu)}{\lambda+2\mu}\tfrac{1}{r}g^{'} = -\tfrac{2\rho\Omega^2\beta}{\lambda+2\mu}r\ , \\ g^{''} + \tfrac{2}{r}g^{'} - \tfrac{6(\lambda+2\mu)}{\mu}\tfrac{1}{r^2}g + \tfrac{2(\lambda+2\mu)}{\mu}\tfrac{1}{r^2}f_2 + \\ +\tfrac{\lambda+\mu}{\mu}\tfrac{1}{r}f_2^{'} = -\tfrac{\rho\Omega^2\beta}{\mu}r\ .\end{gathered} \tag{13.98}$$

We seek solutions of these equations of the form $f_{1,2}$, $g = Ar^n$; the solution of the homogeneous equations (regular in the origin) corresponds to $n = 1$; we get

$$f_1 = -\frac{\rho\Omega^2\alpha}{10(\lambda+2\mu)}r^3 + C_1 r \tag{13.99}$$

and

$$f_2 = C_2 r \ , \ g = \frac{\rho\Omega^2\beta}{6\lambda}r^3 + C_3 r \ , \tag{13.100}$$

where $C_{1,2,3}$ are constants of integration. These constants are determined from the boundary conditions given by equations (13.54) for a free surface. Finally, we get the displacement

$$\begin{aligned}\mathbf{u} = -\tfrac{\rho\Omega^2}{3\lambda}\{\tfrac{\lambda}{5(\lambda+2\mu)}r\left(r^2 - \tfrac{5\lambda+2\mu}{3\lambda+2\mu}R^2\right) - \\ -R^2 r(1-3\cos^2\theta)\}\mathbf{e}_r + \\ +\tfrac{\rho\Omega^2}{3\lambda}r\{r^2 - \tfrac{2(3\lambda+\mu)}{3\lambda}R^2\}\sin\theta\cos\theta\mathbf{e}_\theta \ .\end{aligned} \tag{13.101}$$

It is worth estimating the equatorial displacement ($\theta = \pi/2$) for the Earth radius $R = 6370km$; with $\rho = 5g/cm^3$ and λ, $\mu = 10^{11}dyn/cm^2$ we get $u = u_r \simeq 10km$.

13.2.8 Earthquake "temperature"

Let us multiply by $\dot{\mathbf{u}}$ the equation of the elastic motion,

$$\rho\ddot{\mathbf{u}} + \mu curl\, curl\mathbf{u} - (\lambda+2\mu)grad\, div\mathbf{u} = \mathbf{F} \ ; \tag{13.102}$$

integrating by parts, we get the law of energy conservation

$$\frac{\partial\mathcal{E}}{\partial t} = -div\mathbf{S} + w \ , \tag{13.103}$$

where

$$\mathcal{E} = \frac{1}{2}\rho\dot{\mathbf{u}}^2 + \frac{1}{2}\mu(curl\mathbf{u})^2 + \frac{1}{2}(\lambda+2\mu)(div\mathbf{u})^2 \tag{13.104}$$

is the energy density,

$$S_i = \mu(\dot{u}_j \partial_j u_i - \dot{u}_j \partial_i u_j) - (\lambda + 2\mu)\dot{u}_i \partial_j u_j \qquad (13.105)$$

are the components of the energy flux density and $w = \dot{\mathbf{u}}\mathbf{F}$ is the density of mechanical work done by the external force per unit time. It is worth noting that the energy density given by equation (13.104) differs from the energy density derived from the other form of the equation of motion, *e.g.*,

$$\rho\ddot{\mathbf{u}} - \mu\Delta\mathbf{u} - (\lambda + \mu) grad\, div \mathbf{u} = \mathbf{F} \ , \qquad (13.106)$$

by the divergence of a vector; it follows that the energy density and the energy flux density are not unique (well defined).

Making use of equations (13.43), (13.44) and (13.47) we can write symbolically

$$\begin{gathered} curl\mathbf{u} = \tfrac{h}{r} l(l+1)\mathbf{R} + \tfrac{1}{r}\tfrac{d}{dr}(rh)\mathbf{S} + \left[\tfrac{f}{r} - \tfrac{1}{r}\tfrac{d}{dr}(rg)\right]\mathbf{T} \ , \\ div\mathbf{u} = \tfrac{1}{r^2}\tfrac{d}{dr}(r^2 f) - \tfrac{g}{r} l(l+1) \ . \end{gathered} \qquad (13.107)$$

We compute the total energy E by introducing these expressions for $curl\mathbf{u}$ and $div\mathbf{u}$ in equation (13.104), integrating over the solid angle and integrating by parts over the radius r; for large values of R the boundary conditions given by equations (13.54) for free vibrations ensure the vanishing of the "surface terms" in the r-integration by parts; in addition, for large values of R we may neglect the f-term in $curl\mathbf{u}$ and the g-term in $div\mathbf{u}$; making use of the equations of motion (13.49), we get finally

$$E \simeq \frac{2l+1}{8\pi} \int d\mathbf{r} \rho\omega^2 \left[f^2 + l(l+1)g^2 + l(l+1)h^2\right] \ , \qquad (13.108)$$

where the summation over l is omitted (the factor $2l+1$ arises from the summation over m). The functions ωf, ωg and ωh in equation (13.108) are superpositions of their own normal modes (labelled by n); for large values of R all these eigenmodes may be taken as the spherical Bessel functions and the eigenfrequencies are given by the zeros of the derivatives of the spherical Bessel functions; we note that these eigenmodes are orthogonal with respect to the r-integration; the f-part in equation (13.108) is related to the velocity c_1 (the combination of $\lambda + 2\mu$ of the elastic moduli), while the g- and h-parts are related to the velocity c_2 (modulus μ).

Let us write the energy given by equation (13.108) for the normal modes as

$$E \simeq \frac{1}{8\pi} \sum_{lmn} \int d\mathbf{r} \left[\rho\omega_{ln}^{(r)2} f_{ln}^2 + \rho\omega_{ln}^{(s)2} g_{ln}^2 + \rho\omega_{ln}^{(t)2} h_{ln}^2 \right] \ , \qquad (13.109)$$

where the summation over m is restored and the coefficients $l(l+1)$ are included in g_{ln} and h_{ln}. We may use approximately the asymptotic expressions for the functions f_{ln}, g_{ln}, h_{ln} of the form $f_{ln} = a_{ln} \cos[kr - (l+1)\pi/2]/kr$ (spherical Bessel functions), with amplitudes a_{ln}; and, similarly, for g_{ln} and h_{ln} with amplitudes b_{ln} and c_{ln}. Effecting the integral, we get

$$E \simeq \frac{1}{4} R \sum_{lmn} \left[\rho c_1^2 a_{ln}^2 + \rho c_2^2 b_{ln}^2 + \rho c_2^2 c_{ln}^2 \right] \ , \qquad (13.110)$$

where R is the radius of the sphere and $c_{1,2}$ are the wave velocities. This is a simple expression, of the form

$$E = \sum_s \rho R c^2 a_s^2 \ , \qquad (13.111)$$

where s is a generic notation for the normal modes.

Let us assume that energy E is given to the vibrating sphere; we ask how it is distributed among the normal modes. It is reasonable to assume that, after many reflections from the surface, the distribution of energy reaches an equilibrium state, in the sense that it does not depend anymore on the time. This state is characterized by a probability density w, which is multiplicative for different spheres; $\ln w$ is additive and function

$$S = -w \ln w \qquad (13.112)$$

should have a maximum value in the equilibrium state, corresponding to a maximal "disorder"; this represents our idea of equilibrium. Obviously, the function S given by equation (13.112) is the entropy.

Its maximum value for constant energy is reached for the extremum of the function $S-\beta wE$, where β is a Lagrange multiplier; we get the Boltzmann (canonical) distribution

$$w = const \cdot e^{-\beta E} \ , \tag{13.113}$$

or, for one mode,

$$w = \sqrt{\beta\rho Rc^2/\pi} e^{-\beta\rho Rc^2 a^2} \ . \tag{13.114}$$

The mean energy per mode is

$$\overline{e} = \frac{1}{2}\sqrt{\rho Rc^2 T} \tag{13.115}$$

and the mean value of the square amplitude is

$$\overline{a^2} = \frac{1}{2}\sqrt{\frac{T}{\rho Rc^2}} \ , \tag{13.116}$$

where we introduced the temperature $T = 1/\beta$. The total mean energy is $\overline{E} = N\overline{e} = N\sqrt{\rho Rc^2 T}/2$, where N is the total number of modes; this equality gives the temperature parameter.

Making use of the asymptotic expressions of the spherical Bessel functions (for the radial functions) we get the normal modes given by $k_{ln}R = (2n+l+1)\pi/2$; hence, we see that the normal modes are equidistant; the corresponding wavelengths are $\lambda_{ln} = 4R/(2n+l+1)$. We may take, tentatively, a cutoff of short wavelengths of the order $1cm$ (corresponding to a frequency $\simeq 500kHz$, for velocity $5km/s$); it is reasonable to admit that below this distance the homogeneous elastic qualities of the Earth do no hold anymore. For this cutoff, we get a maximum number $2n+l+1$ of the order $N_c = 10^9$ and a number of modes of the order $N = N_c^3 = 10^{27}$. For an energy $\overline{E} = 10^{26} dyn \cdot cm$ (corresponding to an earthquake of magnitude $M_w = 7$) we get, from equation (13.115), a temperature $T = 10^{-22} erg$ ($i.e.,\simeq 10^{-5}K$, since $1.38 \times 10^{17}K = 1erg = 1dyn \cdot cm$); the quantity ρRc^2 in equation (13.115) is $\rho Rc^2 \simeq 10^{20} g/s^2$ (for $\rho = 5g/cm^3$, $R \simeq 6 \times 10^8 cm$ and $c = 5km/s$). The estimation of the temperature is very sensitive to the number of eigenmodes N; for instance, for a cutoff wavelength

$10cm$ we get a temperature $T \simeq 10K$. Part of the energy released in an earthquake is spent in mechanical work associated with the motion of the rocks, soil and the damage produced at the Earth's surface; the remaining is dissipated as heat, after a long while; we may see that a big earthquake ($M_w = 7$) may raise the Earth's temperature by as much as cca $10^{-5}K - 10K$ (the inner Earth's temperature is $\simeq 6000K$). We note that the cutoff wavelength, which affects essentially the numerical estimation of the temperature, corresponds to the mean inter-atomic distance in the Debye estimation of the statistical equilibrium of the elastic vibrations (phonons) in a crsytal.

Finally, we note that apart from self-gravitation and rotation, the inhomogeneities may bring an important effect upon the vibrations of the solid sphere. For instance, from equation (13.38), a (uniform) change $\delta\rho$ in density cause a change $\delta\omega/\omega = -\delta\rho/2\rho$ in frequency. The effect of similar changes in the elastic moduli λ and μ can be estimated by using the changes in the wave velocities c in the relation $\omega_{ln} \simeq (c/R)\beta_{ln}$.[41]

[41] K. Aki and P. G. Richards, *Quantitative Seismology*, University Science Books, Sausalito, CA (2009).

14 Two Dimensions

14.1 Seismic waves in two dimensions

A two-dimensional problem of elastic motion views the elastic solid as a three-dimensional solid whose motion does not depend on a coordinate, say z. The Navier-Cauchy equation depends on time and only on two coordinates, say x and y. This is an unphysical situation, which may lead to divergences. For instance, the force is not placed in a point, but along a line (along coordinate z, a line load). The interest for such a problem resides in its somewhat simplifying calculations, though the divergent solutions make it useless.

Let us consider the equation of elastic waves in a three-dimensional solid with a tensorial force localized at $\boldsymbol{R} = 0$ is

$$\ddot{\boldsymbol{u}} - c_t^2 \Delta \boldsymbol{u} - (c_l^2 - c_t^2) grad\, div \boldsymbol{u} = \boldsymbol{F} \ , \tag{14.1}$$

where $\boldsymbol{u}$ is the displacement, $c_{l,t}$ are the longitudinal and transverse velocities,

$$F_i = m_{ij} T \delta(t) \partial_j \delta(\boldsymbol{R}) \tag{14.2}$$

are the force components (per unit mass), m_{ij} is the (symmetric) tensor of the moment per unit mass and T is the duration of the force; the cartesian indices are i, $j = x, y, z$.[1] The tensor m_{ij} is $m_{ij} = M_{ij}/\rho$, where M_{ij} is the moment tensor and ρ is the density of the body. Let us distribute uniformly the source along the z-axis at points of coordinates z_a, denote the solutions by u_a and sum up the equations:

$$\begin{gathered} \sum_a \ddot{u}_{ia} - c_t^2 \sum_a \Delta u_{ia} - (c_l^2 - c_t^2) \sum_a grad_i\, div \boldsymbol{u}_a = \\ = m_{ij} T \delta(t) \partial_j \delta(\boldsymbol{r}) \sum_a \delta(z - z_a) \ , \end{gathered} \tag{14.3}$$

[1] B. F. Apostol, *The Theory of Earthquakes*, Cambridge International Science Publishing, Cambridge (2017); *Introduction to the Theory of Earthquakes*, Cambridge International Science Publishing, Cambridge (2017); *Seismology*, Nova, NY (2020).

where $\boldsymbol{R} = (\boldsymbol{r}, z)$. We may pass to integration in the a-summation, and see that the equation becomes a two-dimensional equation

$$\ddot{v}_i - c_t^2 \Delta v_i - (c_l^2 - c_t^2) grad_i\, div \boldsymbol{v} = m_{i\alpha} T \delta(t) \partial_\alpha \delta(\boldsymbol{r}) \ , \tag{14.4}$$

where $\boldsymbol{v} = \int dz' \boldsymbol{u}(\boldsymbol{r}, z - z')$ and $\alpha = x, y$ (and the div is two-dimensional). This is a cylindrical equation, with solution $\int dz \boldsymbol{u}(\boldsymbol{r}, z)$. We note that integrating equation (14.1) with respect to z leads to a solution which exists for all z but does not depend on z. Alternatively, we may replace the function $\delta(z)$ on the right in equation (14.1) by $1/d$, where d is a small cutoff thickness of a source, and get a two-dimensional equation with the surface density ρd in the denominator on the right. In the two-dimensional limit $\rho d \to 0$ we get an infinite right-side term of the equation. We may keep d small but finite, but this introduces an uncertainty of the order d in the position $\boldsymbol{r}$. It may be taken as the dimension of the, otherwise undetermined, two-dimensional source. Rigorously speaking the equation of the elastic waves in two dimensions is unphysical, and a cutoff procedure should be employed in order to give sense to solution. A similar situation occurs also in one dimension (although, in contrast to the one-dimensional case, in two dimensions the wavefronts are divergent). These equations and their physical parameters are, in general, different from the wave equations in thin rods and plates (shells), or strings and membranes.[2]

For $i = 3$ equation (14.1) reads

$$\ddot{u}_3 - c_t^2 \Delta u_3 = \frac{m_{3\alpha}}{d} T \delta(t) \partial_\alpha \delta(\boldsymbol{r}) \ , \tag{14.5}$$

with solution

$$\begin{aligned} u_3 = \tfrac{T m_{3\alpha}}{2\pi c_t d} \int d\boldsymbol{r}' \tfrac{\theta(ct - |\boldsymbol{r} - \boldsymbol{r}'|)}{\sqrt{c^2 t^2 - (\boldsymbol{r} - \boldsymbol{r}')^2}} \partial_\alpha \delta(\boldsymbol{r}') = \\ = \tfrac{T m_{3\alpha}}{2\pi c_t d} \partial_\alpha \tfrac{\theta(ct - r)}{\sqrt{c^2 t^2 - r^2}} \ . \end{aligned} \tag{14.6}$$

[2] L. Landau and E. Lifshitz, *Course of Theoretical Physics*, vol. 7, *Theory of Elasticity*, Elsevier, Oxford (1986); A. N. Tikhonov and A. A. Samarskii, *Equations of Mathematical Physics*, Dover, NY (1963).

Indeed, the Green function of the propagating-wave problem in two dimensions, *i.e.* the solution of the equation

$$\ddot{G} - c^2 \Delta G = \delta(t)\delta(\boldsymbol{r}) \tag{14.7}$$

can be obtained immediately by Fourier transformations and by placing the poles in the lower ω-plane, according to the causality principle; also, we need the Weyl-Sommerfeld integrals[3]

$$\begin{aligned} \int_0 dx J_0(x) \cos \lambda x &= \tfrac{\theta(1-\lambda)}{\sqrt{1-\lambda^2}} \ , \\ \int_0 dx J_0(x) \sin \lambda x &= \tfrac{\theta(\lambda-1)}{\sqrt{\lambda^2-1}} \end{aligned} \tag{14.8}$$

$(\lambda > 0)$. The Green function is

$$G(t, \boldsymbol{r}) = \frac{\theta(ct - r)}{2\pi c\sqrt{c^2t^2 - r^2}} \ . \tag{14.9}$$

The derivative $\partial_\alpha \frac{1}{\sqrt{c^2t^2-r^2}}$ is a solution of the free equation, such that, according to the regularization procedure, we retain only

$$u_3 = -\frac{Tm_{3\alpha}n_\alpha}{2\pi c_t d} \frac{\delta(ct-r)}{\sqrt{c^2t^2 - r^2}} \ , \tag{14.10}$$

where $\boldsymbol{n} = \boldsymbol{r}/r$. This solution is divergent, due to the factor $1/\sqrt{c^2t^2 - r^2}$. According to the above discussion, we may replace $\sqrt{c^2t^2 - r^2}$ for $ct = r$ by $\sqrt{rd}$ (for any fixed t). (Also, $\delta(ct-r)$ may be viewed as $1/d$ for $ct = r$). Therefore, we may represent this solution as

$$u_3 = -\frac{Tm_{3\alpha}n_\alpha}{2\pi c_t d\sqrt{d}} \frac{\delta(ct-r)}{\sqrt{r}} \ . \tag{14.11}$$

Let us solve the remaining equations (14.1)

$$\ddot{u}_\alpha - c_t^2 \partial_\beta \partial_\beta u_\alpha - (c_l^2 - c_t^2)\partial_\alpha \partial_\beta u_\beta = f_\alpha \delta(t)\delta(\boldsymbol{r}) \ , \tag{14.12}$$

where $f_\alpha = \frac{Tm_{\alpha\beta}}{d}\partial_\beta$. We introduce the Helmholtz potentials Φ and $\boldsymbol{A} = (0, 0, A)$ by

$$u_\alpha = \partial_\alpha \Phi + \varepsilon_{\alpha\beta 3}\partial_\beta A \ , \tag{14.13}$$

[3] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, 6th ed., Academic, NY (2000), p. 709, 6.671 (1,2).

where α, $\beta = 1(x)$, $2(y)$,$3 = z$ and $\varepsilon_{\alpha\beta 3}$ is the totally antisymmetric tensor of rank three; we can see that the second term on the right in this equation is a *curl*. Similarly, we write

$$f_\alpha \delta(t)\delta(\boldsymbol{r}) = \partial_\alpha \varphi + \varepsilon_{\alpha\beta 3}\partial_\beta h \ , \tag{14.14}$$

such that equations (14.12) are split in two equations

$$\ddot{\Phi} - c_l^2 \Delta \Phi = \varphi \ , \ \ddot{A} - c_t^2 \Delta A = h \ , \tag{14.15}$$

where the potentials φ and h are given by

$$\begin{gathered} \Delta\varphi = \left[f_\alpha \partial_\alpha \delta(\boldsymbol{r})\right]\delta(t) \ , \\ \Delta h = \varepsilon_{3\alpha\beta}\left[f_\alpha \partial_\beta \delta(\boldsymbol{r})\right]\delta(t) \ . \end{gathered} \tag{14.16}$$

The solutions of these Poisson equations are obtained by making use of the Green function $\frac{1}{2\pi}\ln(r)$ of the laplacian in two dimensions. We get immediately

$$\begin{gathered} \varphi = \tfrac{1}{2\pi}\delta(t) f_\alpha \partial_\alpha \ln(r) \ , \\ h = \tfrac{1}{2\pi}\delta(t)\varepsilon_{3\alpha\beta} f_\alpha \partial_\beta \ln(r) \end{gathered} \tag{14.17}$$

and equations (14.15) become

$$\begin{gathered} \ddot{\Phi} - c_l^2 \Delta\Phi = \tfrac{1}{2\pi}\delta(t) f_\alpha \partial_\alpha \ln(r) \ , \\ \ddot{A} - c_t^2 \Delta A = \tfrac{1}{2\pi}\delta(t)\varepsilon_{3\alpha\beta} f_\alpha \partial_\beta \ln(r) \ . \end{gathered} \tag{14.18}$$

These equations lead to the functions $F_{l,t}$ given by

$$\ddot{F} - c^2 \Delta F = \delta(t)\ln(r) \ ; \tag{14.19}$$

indeed, $\Phi = \frac{1}{2\pi} f_\alpha \partial_\alpha F_l$ and $A = \frac{1}{2\pi}\varepsilon_{3\alpha\beta} f_\alpha \partial_\beta F_t$, such that the solution can be represented as

$$u_\alpha = \frac{1}{2\pi} f_\alpha \partial_\beta \partial_\beta F_t + \frac{1}{2\pi} f_\beta \partial_\alpha \partial_\beta (F_l - F_t) \ . \tag{14.20}$$

The solution of equation (14.19) can be written as

$$\begin{gathered} F = \tfrac{1}{2\pi c}\int d\boldsymbol{r}' \frac{\theta(ct - |\boldsymbol{r}-\boldsymbol{r}'|)}{\sqrt{c^2t^2 - (\boldsymbol{r}-\boldsymbol{r}')^2}} \ln(r') = \\ = \tfrac{1}{2\pi c}\int d\boldsymbol{r}' \frac{\theta(ct - r')}{\sqrt{c^2t^2 - r'^2}} \ln \mid \boldsymbol{r} - \boldsymbol{r}' \mid \ . \end{gathered} \tag{14.21}$$

By using an integration by parts, we get

$$F = t\ln(r) + \tfrac{1}{2c}\int_0^{ct} dr' \tfrac{\sqrt{c^2t^2 - r'^2}}{r'} + \\ + \tfrac{1}{4\pi c}\int_0^{ct} dr' \tfrac{\sqrt{c^2t^2 - r'^2}}{r'}(r'^2 - r^2)\int d\varphi \tfrac{1}{r^2 + r'^2 - 2rr'\cos\varphi} \ . \tag{14.22}$$

The φ-integral is a Poisson integral; it can be performed by the substitution $\tan\frac{\varphi}{2} = t$; we get

$$\int d\varphi \frac{1}{r^2 + r'^2 - 2rr'\cos\varphi} = \frac{2\pi}{\mid r^2 - r'^2 \mid} \tag{14.23}$$

and

$$F = t\ln(r) + \tfrac{1}{2c}\int_0^{ct} dr' \tfrac{\sqrt{c^2t^2 - r'^2}}{r'} - \\ - \tfrac{1}{2c}\int_0^{ct} dr' \tfrac{\sqrt{c^2t^2 - r'^2}}{r'} sgn(r - r') \ . \tag{14.24}$$

We need the spatial derivatives of the functions $F_{l,t}$ in order to get the solution u_α. We note that $t\ln r$ is a solution of the free wave equation, so, according to the regularization procedure, we leave this term aside (indeed, it is unphysical, because it lacks an adequate cutoff). Next, we notice that the second term on the right in equation (14.24) does not depend on r, so it does not contribute to the spatial derivatives of the function F; we leave it aside. Then, although we can do the remaining integral, we notice that we need it not: we need only its spatial derivatives. Consequently, we get from equation (14.24)

$$\frac{\partial F}{\partial r} = -\frac{1}{c}\frac{\sqrt{c^2t^2 - r^2}}{r}\theta(ct - r) \ . \tag{14.25}$$

Indeed, by taking the laplacian in equation (14.19), we can see that ΔF is the Green function of the wave equation, $\Delta F = \frac{1}{c}\frac{\theta(ct-r)}{\sqrt{c^2t^2 - r^2}}$; by using $\Delta = \frac{1}{r}\frac{d}{dr}(r\frac{d}{dr})$, we can integrate this equation, and the result is precisely that given by equation (14.25).

We introduce the notations $u_{1\alpha} = \frac{1}{2\pi} f_\alpha \partial_\beta \partial_\beta F$ and $u_{2\alpha} = \frac{1}{2\pi} f_\beta \partial_\alpha \partial_\beta F$ (equation (14.20)). Making use of the results obtained above, we get

$$\boldsymbol{u}_1 = \tfrac{T}{2\pi cd}\boldsymbol{m}\tfrac{r}{(c^2t^2 - r^2)^{3/2}}\theta(ct - r) - \\ - \tfrac{T}{2\pi cd}\boldsymbol{m}\tfrac{1}{(c^2t^2 - r^2)^{1/2}}\delta(ct - r) \tag{14.26}$$

and

$$\boldsymbol{u}_2 = \frac{T}{2\pi cd}[(2\boldsymbol{m} + m_0\boldsymbol{n})\frac{2c^2t^2-r^2}{r^3(c^2t^2-r^2)^{1/2}} -$$

$$-m_4\boldsymbol{n}\frac{8c^4t^4-12c^2t^2r^2+3r^4}{r^3(c^2t^2-r^2)^{3/2}}]\theta(ct-r)- \quad (14.27)$$

$$-m_4\boldsymbol{n}\frac{1}{(c^2t^2-r^2)^{1/2}}\delta(ct-r) \ ,$$

where $m_0 = m_{\alpha\alpha}$, $\boldsymbol{m}$ is the vector with components $m_\alpha = m_{\alpha\beta}n_\beta$, $m_4 = \boldsymbol{mn}$ and $\boldsymbol{n} = \boldsymbol{r}/r$. The solution is divergent at the wavefront $ct = r$. According to the discussion made above we replace $1/(c^2t^2 - r^2)$ for $ct = r$ by $1/rd$. The θ-terms contribute to the near-field zone ($r \ll ct$), while the δ-terms can be associated to the far-field zone (wavefront); both terms have the same kind of divergences. The full δ-part of the solution is

$$\boldsymbol{u}_\delta = -\frac{T}{2\pi c_l d}\boldsymbol{m}\frac{1}{(c_l^2t^2-r^2)^{1/2}}\delta(c_lt-r)-$$
$$-\frac{T}{2\pi c_t d}(\boldsymbol{m}-m_4\boldsymbol{n})\frac{1}{(c_t^2t^2-r^2)^{1/2}}\delta(c_tt-r) \ . \quad (14.28)$$

We can see that the velocity c_l is associated to longitudinal waves, while the velocity c_t is associated to transverse waves. This property is not shared by the θ-terms. The structure of the solution is very similar to the structure of the solution in three dimensions.

14.2 Seismic main shock in two dimensions

Once arrived at the plane surface of a half-space, the seismic waves generate a boundary force which, according to Huygens's principle, is the source of secondary waves. These waves are the seismic main shock on the surface. The wavefronts of the seismic waves intersect the plane surface along two circles, corresponding to their velocities $c_{l,t}$, which increase their radii with velocities $v_{l,t}$ greater than $c_{l,t}$. Along these circles wave sources appear, which are proportional to $\delta(r - v_{l,t}t)\delta(z)$, for the Helmhotz potentials, where r is the distance from the epicentre, z is the coordinate perpendicular to the surface and the time t is measured from the moment the waves arrive at the epicentre. The secondary waves generated by such sources go like

$(c_{l,t}^2 t^2 - r^2)^{-3/2}$ for $c_{l,t}t > r$ and 0 for $c_{l,t}t < r$ on the surface. We can see that we have two wall-like main shocks, one propagating with velocity c_l (radial component) and another propagating with velocity c_t (angular and vertical components); the calculations are valid for a free surface and $v_{l,t} \simeq c_{l,t}$.[4]

In two dimensions the waves intersect the surface line in two points, so the source is proportional to $\delta(\mid x \mid - v_{l,t}t)\delta(z)$. The secondary waves are obtained by using the Green function given by equation (14.9) for such a source. We are led to the integral

$$I = \int dx' \frac{\theta\left[c(t - \mid x' \mid /v) - \sqrt{(x-x')^2 + z^2}\right]}{\sqrt{c^2(t - \mid x' \mid /v)^2 - [(x-x')^2 + z^2]}} \ . \tag{14.29}$$

For $v \simeq c$ this integral is

$$I = -\frac{\sqrt{c^2t^2 - R^2}}{\mid\mid x \mid - ct \mid} \ , \ ct > R \ , \tag{14.30}$$

and 0 otherwise, where $R = \sqrt{x^2 + z^2}$. On the surface line ($z = 0$) the spatial derivatives of this integral give displacements proportional to $(ct - \mid x \mid)^{-3/2}$ for $ct > \mid x \mid$ and zero for $ct < \mid x \mid$. The main shock(s) in two dimensions is (are) similar to the main shock(s) in three dimensions.

14.3 Vibrations in two dimensions

A convenient means for the Navier-Cauchy equation in two dimensions is the orthogonal vector plane waves

$$\boldsymbol{Z}(k) = \boldsymbol{e}_z \frac{e^{ikx}}{\sqrt{2\pi}} \ , \ \boldsymbol{G}(k) = i\boldsymbol{e}_x \frac{e^{ikx}}{\sqrt{2\pi}} \ , \tag{14.31}$$

constructed by analogy to the vector plane-wave functions in three dimensions, where $\boldsymbol{e}_{x,z}$ are the unit vectors along the x, z-directions. For vibrations the Navier-Cauchy equation reads

$$c_2^2 curl\, curl \boldsymbol{u} - c_1^2 grad\, div \boldsymbol{u} - \omega^2 \boldsymbol{u} = \boldsymbol{F} \ , \tag{14.32}$$

[4]See, for instance, B. F. Apostol, *Seismology*, Nova, NY (2020).

where $c_{1,2}$ are the velocities of the elastic waves, ω is the frequency and $\boldsymbol{F}$ is the force per unit mass. We use the expansions

$$\boldsymbol{u} = \int dk(f\boldsymbol{Z} + g\boldsymbol{G}) \ , \ \boldsymbol{F} = \int dk(F_z\boldsymbol{Z} + F_g\boldsymbol{G}) \tag{14.33}$$

for a seismic tensorial force

$$\boldsymbol{F}_i = m_{ij}\partial_j\delta(\boldsymbol{r} - \boldsymbol{r}_0) \tag{14.34}$$

placed at $\boldsymbol{r}_0$, where $m_{ij} = m_{ij}(\omega)$ is the Fourier transform of the seismic moment per unit mass and $i, j = x, z$. We choose a half-plane $z < 0$, where $\boldsymbol{r}_0 = (0, z_0)$, $z_0 < 0$, and an isotropic moment $m_{ij} = -m\delta_{ij}$. Equation (14.32) becomes

$$\begin{aligned} c_1^2 f'' + (\omega^2 - c_2^2k^2)f - (c_1^2 - c_2^2)kg' &= \tfrac{m}{\sqrt{2\pi}}\delta'(z - z_0) \ , \\ c_2^2 g'' + (\omega^2 - c_1^2k^2)g + (c_1^2 - c_2^2)kf' &= \tfrac{mk}{\sqrt{2\pi}}\delta(z - z_0) \ , \end{aligned} \tag{14.35}$$

where the derivatives are taken with respect to the variable z. We can see that the structure of these equations are the same as in three dimensions, as expected. We follow the analysis done in three dimensions and get the solutions

$$\begin{aligned} f = &\tfrac{(\kappa_2^2 - k^2)^2 - 4\kappa_1\kappa_2k^2}{\Delta}\tfrac{m\sqrt{2\pi}}{4c_1^2}e^{-i\kappa_1(z+z_0)} + \\ &+\tfrac{(\kappa_2^2 - k^2)k^2}{\Delta}\tfrac{m\sqrt{2\pi}}{c_1^2}e^{-i\kappa_1z_0 - i\kappa_2z} + \\ &+sgn(z - z_0)\tfrac{m\sqrt{2\pi}}{4c_1^2}e^{i\kappa_1|z-z_0|} + c.c \end{aligned} \tag{14.36}$$

and

$$\begin{aligned} g = &\tfrac{i\left[(\kappa_2^2 - k^2)^2 - 4\kappa_1\kappa_2k^2\right]k}{\kappa_1\Delta}\tfrac{m\sqrt{2\pi}}{4c_1^2}e^{-i\kappa_1(z+z_0)} - \\ &-i\tfrac{(\kappa_2^2 - k^2)\kappa_2k}{\Delta}\tfrac{m\sqrt{2\pi}}{c_1^2}e^{-i\kappa_1z_0 - i\kappa_2z} - \\ &-i\tfrac{m\sqrt{2\pi}k}{4\kappa_1c_1^2}e^{i\kappa_1|z-z_0|} + c.c \ , \end{aligned} \tag{14.37}$$

where

$$\Delta = (\kappa_2^2 - k^2)^2 + 4\kappa_1\kappa_2k^2 \tag{14.38}$$

and

$$\kappa_{1,2} = \sqrt{\omega^2/c_{1,2}^2 - k^2} \ ; \qquad (14.39)$$

we use the convention $\kappa_{1,2} \rightarrow i\kappa_{1,2}$ for $\kappa_{1,2}^2 = k^2 - \omega^2/c_{1,2}^2 > 0$.
As it is well known from the analysis in three dimensions the determinant Δ has two zeros $\omega_s = \pm c_2\xi_0 \mid k \mid$, where ξ_0 varies between 0.87 and 0.95 (it depends on the ratio c_2/c_1, which varies between $1/\sqrt{2}$ and 0). We note that this solution corresponds to damped waves ($\kappa_{1,2} \rightarrow i\kappa_{1,2}$). In the vicinity of these zeros we have

$$\Delta \simeq \frac{k^2}{c_2^2\sqrt{1-\xi_0^2}}(\omega^2 - \omega_s^2) \ , \qquad (14.40)$$

where we use the (numerical) approximations $\xi_0 \simeq 1$ and $c_2^2/c_1^2 \ll 1$, wherever appropriate. In equation (14.33) we perform first the ω-integration. This integration is given by the residues f_s, g_s in the poles generated by $\Delta = 0$. For $m(t) = T\delta(t)$, $x = 0$ and the numerical approximations above, we get

$$f_s(k) = \sqrt{2\pi(1-\xi_0^2)}\frac{mc_2T}{2c_1^2} \mid k \mid \sin c \mid kt \mid e^{-|kz_0|} \qquad (14.41)$$

and the component

$$u_z = -\sqrt{1-\xi_0^2}\tfrac{mc_2T}{c_1^2}\cdot$$
$$\cdot\tfrac{\partial}{\partial|z_0|}Re\int_0^{+\infty} dk \sin ck \mid t \mid e^{-k|z_0|}e^{ikx} \qquad (14.42)$$

of the displacement, where $c = c_2\xi_0$. The intervening integral is

$$I = \int_0^{+\infty} dk \sin ck \mid t \mid e^{-k|z_0|}e^{ikx} =$$
$$= \tfrac{c|t|}{\left[(z_0^2 - x^2 + c^2t^2)^2 + 4x^2z_0^2\right]^{1/2}}e^{i\chi} \ , \qquad (14.43)$$

where

$$\tan\chi = \frac{2x \mid z_0 \mid}{z_0^2 - x^2 + c^2t^2} \ . \qquad (14.44)$$

This integral is vanishing for $\mid x \mid \rightarrow \infty$ for any moment of time, as well as for any position x and $\mid t \mid \rightarrow \infty$; it exhibits a bump placed at

$x_0 = \pm\sqrt{z_0^2 + c^2t^2}$, of the order $c \mid t \mid /x_0^2 \mid z_0 \mid$, which propagates with velocity $\pm c$. This behaviour is specific to a vibration, which affects the whole surface line, the bump arising from the temporal-impulse character of the force, which requires, in fact, a propagating-wave treatment. A similar behaviour is valid for the component u_x.

The vector plane waves given by equation (14.31) can also be used for the propagating-wave problem; in two dimensions it is treated exactly the same as in three dimensions, leading to the results given above (we should perform simultaneously the double integral with respect to k and q, and get a Weyl-Sommerfeld integral).

15 A Critical History of Seismology

(contributed by Marian Apostol)

15.1 Elasticity

Earthquakes are known since long. Sometimes, in some places, the ground begins suddenly to shake; this motion ceases in a relatively short time, though small tremors may be felt long after the shaking. Something, a more or less violent motion occurs beneath the Earth's surface, and this motion is transmitted to the surface. The first great advance in understanding such motion was made by Hooke, around 1660, who realized that bodies are elastic, and the elastic force is proportional to the displacement (elongation, compression): *ut tensio, sic vis*. On this discovery, around $1821-22$, Navier and Cauchy built up the mathematical theory of elastic continuous media, known as the Navier-Cauchy equation. This was a great advance in science. Around the same time Poisson showed that in homogeneous and isotropic elastic solids there exist two kinds of elastic waves, one compressional and another transverse. In 1848 Kelvin computed the equilibrium of an elastic body under a point force and in 1849 Stokes computed the waves generated in an elastic body by a point force. In 1880 Lamb computed the vibrations of an elastic sphere and in 1885 Rayleigh discovered the so-called surface waves. At the beginning of the 20th century Love and Stoneley computed the vibrations of an elastic slab and an elastic interface. A series of equilibrium problems of static elasticity have been solved at the end of the 19th century and in the first half of the 20th century by Boussinesq, Cerruti, Flamant, Melan, Mindlin. With these works the development of elasticity ended,[1] although there exist many unsolved problems, in particular the application of the theory of elasticity to Seismology. Such attempts have not been very happy.

15.2 Seismological problem

The great progress in Seismology was made by the invention of the seismograph, at the end of the 19th century by Milne and others.[2] The seismograph is a pendullum, capable of recording local motion at Earth's surface (displacement, velocity, acceleration) in all the three directions. The recordings of the seismograph are called seismograms. Under the action of the motion of the Earth's surface a pendullum starts to move. Thereafter, it follows a combination of the external motion and its own motion (the so-called eigenmodes). From the outset we may see that not all the features of a seismograms are caused by the motion of the ground; some arise from the seismograph own motion. This observation was made early, but it seems it has been forgotten in the modern times. At that time, improving the seismographs was considered a challenge to the modern Seismology.

In 1897 Oldham published an influential paper where he identified on seismograms three types of motion: two faible tremors followed in a short time by a big motion.[3] The two tremors are short, almost structureless, the first one is a longitudinal motion, the second is a transverse motion. Undoubtedly, they are the elastic waves discovered by Poisson and Stokes. They have been called the P (longitudinal) and S (transverse) seismic waves. These waves seem to be spherical-shells. The longitudinal motion in the P wave helped to identify, by

[1]See B. F. Apostol, *The Theory of Earthquakes*, Cambridge International Science Publishers, Cambridge (2017); *Introduction to the Theory of Earthquakes*, Cambridge International Science Publishers, Cambridge (2017); *Seismology*, Nova, NY (2020).

[2]J. Milne, "Notes on the horizontal and vertical motion of the earthquake of March 8, 1881", Trans. Seism. Soc. Japan **3** 129 (1881); Bakerian Lecture, Proc. Roy. Soc. London **A77** 365 (1906).

[3]R. D. Oldham, *Report on the Great Earthquake of 12th June, 1897*, Geol. Surv. India Memoir 29 (1899); "On the propagation of earthquake motion to long distances", Trans. Phil. Roy. Soc. London **A194** 135 (1900).

triangulation, the position of the source of these waves, the earthquake focus: the focus is almost pointlike, placed at depths of a few hundreds of kilometers at most. Since the P and S waves are short in time, it follows that the seismic activity in the focus is short; the earthquake's source is practically localized in time, it is a temporal impulse. This point is often overlooked nowadays. Of course, there may exist structured foci, whose activity lasts more, or such foci may even propagate. But the typical, elementary earthquake has a pointlike focus with a temporal impulse seismic activity.

In 1910, by using various considerations, Reid came up with his rebound theory.[4] This theory claims that in the Earth's crust and mantle (in lithosphere) there are tectonic plates, which may move slightly, one against the other. Such a motion, localized in a fault, generates earthquakes. Therefore, the earthquake foci are shearing faults. In 1974 Kostrov made insightful observations concerning the shear motion in an elastic fault, which passed unnoticed.

With the third, large motion, seen on seismograms, Oldham was not so lucky: he assigned it to Rayleigh surface waves. This motion, which succeeds the P and S seismic waves, has an abrupt wall, with a long tail and many oscillations. It is called main shock. This assignation is improper. The only waves produced by the focus are the P and S seismic waves. What would the cause of the main shock be? The Rayleigh surface wave is a plane wave moving along the surface and damped along the direction perpendicular to the surface. It satisfies the boundary conditions of a free surface. Consequently, it is a (damped) vibration perpendicular to the surface (or a guided wave). The P and S seismic waves are localized waves, not extended waves, like the plane waves. The P and S seismic waves are localized in time. Consequently, they may not satisfy boundary conditions, because they have not the time, nor the place. Spherical waves localized in time and space (like the P and S waves) can be decomposed in plane waves (by the so-called Weyl-Sommerfeld expansion), but the members of this decomposition (plane waves) are not real waves. The explanation of the seismograms, especially the main shock, becomes the seismological problem.

[4] H. F. Reid, *Mechanics of the Earthquake, The California Earthquake of April 18, 1906*, vol. 2, Carnegie Institution, Washington (1910).

Lamb attacked the seismological problem in 1904, by what looks like a historical seismological error.[5] He used plane-wave expansions for a temporal-impulse source, as for propagating waves, and impose boundary conditions, as for vibrations. This is improper. Moreover, in effecting the intervening integrals, he saw cuts in the complex plane where none exists. Among other unphysical things, he obtained a small main shock existing long before the earthquake begins. The procedure is somewhat reformulated following Knott and Zoeppritz, who, around the same time, computed reflection and refraction coefficients of plane waves, by boundary conditions at the surface.[6] The identification of such reflected or refracted waves in seismograms is one of the main activity nowadays. The seismological problem has been forgotten.

Furthermore, the motion in the Rayleigh surface waves is in the propagation vertical plane, with the magnitude of the two components comparable. The seismograms show a large motion perpendicular to the propagation vertical plane. To the rescue of the surface-wave assignation the Mohorovicic discontinuity between Earth's crust (approximately $70km$ depth) and Earth's mantle (down to $3000km$) was invoked.[7] The crust vibrates like a solid slab, and these vibrations have a large horizontal amplitude, as seen on seismograms. They are the Love waves (vibrations). Moreover, vibrations on interface may occur, as Stoneley showed (Stoneley waves). It remains to have a convincing knowledge of the sharp Mohorovicic discontinuity.

The seismological problem was never solved. One cause is the lack of knowledge of the force acting in the focus. The Stokes solution for a point force is used for a couple of forces, such that the total force should be zero. But a couple of forces has a non-zero angular momentum (torque), so we put a double couple. But the double couple is arbitrary and depends on the reference frame. Therefore, the

[5] H. Lamb, "On the propagation of tremors over the surface of an elastic solid", Phil. Trans. Roy. Soc. (London) **A203** 1 (1904).

[6] C. G. Knott, "Reflection and refraction of elastic waves with seismological applications", Phil. Mag. **48** 64 (1899); K. Zoeppritz, "Ueber Reflection und Durchgang seismischer Wellen durch Unstetigkeitsflaechen", Erdbebenwellen VIIb, Gottingen, Nachrichten **1** 66 (1919).

[7] A. Mohorovicic, "Das Beben von 8 Okt. 1909", Jahrb. Meteorol. Obs. Zagreb **9** 1 (1909).

seismic source depends on the reference frame, which is unacceptable. The seismic force has been established in 2017; it implies the tensor of the seismic moment.[8] In particular the static deformations of a half-space have been computed for this force,[9] as well as the effect of an internal discontinuity. The P and S seismic waves produced by this force come up as scissor-like sphericall shells, localized in time and space, exactly as seen on seismograms. We call them primary waves. Once arrived on Earth's plane surface (where they propagate faster than the elastic waves!), these primary waves leave behind sources of secondary waves, according to Huygens' principle. These secondary waves have the shape of an abrupt wall with a long tail, which subsides slowly (actually, two neighbouring such structures, for displacement components, corresponding to the two P and S waves). This is the main shock (or shocks), propagating on Earth's surface. Its oscillatory structure, which cannot be accounted for, may arise from: structured focus, structured primary waves, reflection on the neighbouring portions of the spherical surface of the Earth, or seismographs' eigenoscillations.

For our distance scale the Earth's surface may be approximated by a plane (and the Earth by a half-space). However, after shorter or longer times, the secondary waves, which propagate in the whole Earth, are reflected by the spherical surface of the Earth, and multiple reflections may appear, so we may expect the setting of a vibration regime. Indeed, normal modes of the spherical Earth, some with long periods, may be identified by seismographs, in accordance with Lamb's theory of vibrations of an elastic sphere.

The calculations for the seismological problem are done for a homogeneous and isotropic half-space. The Earth is inhomogeneous. It is expected that the inhomogeneities affect much the nature of the elastic waves propagating in the Earth. This problem is considered to be a challenge to the modern Seismology. It is not. The problem is irrelevant. Indeed, the localized spherical waves, which are the seismic waves in a homogeneous and isotropic medium, are made of a super-

[8] B. F. Apostol, "Elastic waves inside and on the surface of a half-space", Q. J. Mech. Appl. Math. **70** 289 (2017).

[9] B. F. Apostol, "Elastic displacement in a half-space under the action of a tensor force. General solution for the half-space with point forces", J. Elast. **126** 231 (2017).

position of all the frequencies (wavelengths), with equal weights. Very likely, the inhomogeneities are distributed in size according to a power law, *i.e.* those with a small size are the most numerous. Consequently, we expect an influence on the short-wavelength part of the spherical waves, which means that they may only acquire a fine structure due to the inhomogeneities, but their localized nature is preserved.[10] The roughness of a surface may lead to localized waves.[11] Sub-surface inhomogeneities may resonate and produce local amplification factors.[12] Designing earthquake-resistant buildings on Earth's surface is considered another challenge to the modern Seismology. Here the philosophy is simple. We should build on high-elasticity ground, where the motion may be large but it is transmitted away. On soft soils the damping collects all the energy on the building.

15.3 Inverse problem

The large energies involved by earthquakes made reasonable the introduction of a logarithmic measure of earthquake's size, called magnitude. Though related to earthquake's energy, the magnitude was first defined by the logarithm of the local displacement, in certain conditions. This is the local magnitude, introduced by Gutenberg and Richter.[13] There are many conventions for defining the local magnitude, and many local-magnitude scales. Later, Hanks and Kanamori related the magnitude to the logarithm of the seismic moment.[14] In-

[10] B. F. Apostol, "The effect of the inhomogeneities on the propagation of elastic waves in isotropic bodies", Mech. Res. Commun. **37** 458 (2010; "Scattering of longitudinal waves (sound) by defects in fluids. Rough surface", Centr. Eur. J. Phys. **11** 1036 (2013).

[11] B. F. Apostol, "Elastic waves in a semi-infinite body", Phys. Lett. **A374** 1601 (2010); "Scattering of electromagnetic waves from a rough surface", J. Mod. Optics **59** 1607 (2012); "The effect of surface inhomogeneities on the propagation of elastic waves", J. Elas. **114** 85 (2014).

[12] B. F. Apostol, "Amplification factors in oscillatory motion", Roum. J. Phys. **49** 691 (2004); "A resonant coupling of a localized harmonic oscillator to an elastic medium", Roum. Reps. Phys. **69** 116 (2017).

[13] B. Gutenberg and C. Richter, "Frequency of earthquakes in California", Bull. Seism. Soc. Am. **34** 185 (1944); "Magnitude and energy of earthquakes", Annali di Geofisica **9** 1 (1956) (Ann. Geophys. **53** 7 (2010)).

[14] H. Kanamori, "The energy release in earthquakes", J. Geophys. Res. **82** 2981 (1977); T. C. Hanks and H. Kanamori, "A moment magnitude scale", J. Geo-

deed, it was realized that the seismic force acting in the focus is governed by the tensor of the seismic moment, so, the magnitude defined this way, which is called the moment magnitude, characterizes the earthquake. However, the relation between the seismic moment and the earthquake's energy is lacking. Moreover, the use of the seismic moment for defining the magnitude was favoured by a certain way of estimating the seismic moment as $\mu S d$, where μ is the shear elastic modulus, S is the area of the rupture at Earth's surface and d is the slip of the rupture. It is thought that the ruptures produced by an earthquake at Earth's surface may provide an estimate of the magnitude of the seismic moment.[15] We can see how approximate is such a procedure. Also, by making use of the double-couple computations, it is claimed that the seismic moment may be derived from the form of the seismic waves recorded at Earth's surface.[16] On one side, such calculations depend on the reference frame; on the other, the correct results of computing the seismic waves imply a regularization of the solutions, because the use of auxiliary functions in solving the Navier-Cauchy equation, like potentials, may introduce spurious features.[17] Apart from not being in the public domain, the procedure of wave-form inversion does not pay attention to such precautions.

The derivation of the tensor of the seismic moment from data measured at Earth' s surface is called the inverse problem in Seismology.

The inverse problem of the Seismology was solved in 2019.[18] The

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phys. Res. **84** 2348 (1979).

[15] D. Wells and K. Coppersmith, "New empirical relationships among magnitude, rupture length, rupture width, rupture area, and surface displacement", Bull. Seism. Soc. Am. **84** 974 (1994).

[16] A. M. Dziewonski abd D. L. Anderson, "Preliminary reference earth model", Phys. Earth planet. Inter. **25** 297 (1981); A. M. Dziewonski, T. A. Chou and J. H. Woodhouse, "Determination of earthquake source parameters from waveform data for studies of global and regional seismicity", J. Geophys. Res. **86** 2825 (1981); S. A. Sipkin, "Estimation of earthquake source parameters by the inversion of waveform data: synthetic waveforms", Phys. Earth planet. Inter. **30** 242 (1982); G Ekstrom, M. Nettles and A. M. Dziewonski, "The global CMT project 2004-2010: centroid-moment tensors for 13,017 earthquakes", Phys. Earth Planet. Int. **200-201** 1 (2012).

[17] M. Apostol, "On unphysical terms in the elastic Hertz potentials", Acta Mech. **228** 2733 (2017).

[18] B. F. Apostol, "An inverse problem in seismology: derivation of the seismic source parameters from P and S seismic waves", J. Seismol. **23** 1017 (2019).

seismic-moment tensor was derived from measurements of the P and S seismic waves at Earth's surface. The solution was given in a consistently covariant form. In addition, the relationship $E = \overline{M}/2\sqrt{2}$ has been established, between the energy E of the earthquake and the mean seismic moment $\overline{M} = (M_{ij}^2)^{1/2}$, where M_{ij} is the seismic-moment tensor. The logarithmic connection between energy and magnitude, known also as the Gutenberg Richter law, was suggested long ago.[19] Besides the tensor of the seismic moment other parameters of the seismic focus have been computed, like the size of the focal region, the duration of the seismic activity in the focus and the orientation of the fault. Moreover, from near-field data in the epicentral region, we may infer the seismic moment and, especially, the tendency of the seismic activity in a seismogenic region.[20]

15.4 Statistical Seismology

The number of small-magnitude earthquakes is extremely large. Consequently, a logarithmic law is appropriate. This is (another) Gutenberg-Richter empirical law, which relates the logarithm of the magnitude density of earthquakes to magnitude.[21] Obviously, it is a statistical law. The use of statistical laws may induce the idea of possible prediction of earthquakes. However, a relevant use of statistical laws requires the understanding of their origin, *e.g.* the origin of the probabilities in earthquakes' distributions. Moreover, there exist deviations from the logarithmic Gutenberg-Richter law, like the roll-off effect at small magnitudes. Very likely, such deviations imply correlations, which again require the understanding of the occurrence probability of the earthquakes. The correlations imply the idea of accompanying seismic events, like foreshocks and aftershocks, associated to a main

[19] T. Utsu and A. Seiki, "A relation between the area of aftershock region and the energy of the mainshock" (in Japanese), J. Seism. Soc. Japan **7** 233 (1955); T. Utsu, "Aftershocks and earthquake statistics (I): some parameters which characterize an aftershock sequence and their interaction", J. Faculty of Sciences, Hokkaido Univ., Ser. VII (Geophysics) **3** 129 (1969).

[20] B. F. Apostol, "Seismic moment deduced from quasi-static surface displacement in seismogen zones", J. Theor. Phys. **279** (2017); "Near-field seismic motion: waves, deformations and seismic moment", J. Theor. Phys. **328** (2021).

[21] B. Gutenberg and C. F. Richter, *loc.cit.*

shock. Also, there exists another empirical law, known as Bath's law, which states that the highest aftershock is by 1.2 smaller in magnitude than the main shock.[22] The foreshock and aftershock distributions deviates from the Gutenberg-Richter standard distribution of regular, background earthquakes.[23] Finally, the aftershocks follow an inverse-time law in their distribution, known as the Omori law.[24] The intervention of the time in statistical laws induces the idea of a relation between the earthquakes' energy and the accumulation time of this energy, which is the earthquake's occurrence time.

Such problems have never been recognized as important problems in Seismology.

"We do not have yet a physical theory regarding the processes that take place at and near the earthquake source, neither prior to the event nor even at the time of its occurrence. ... So, just sprinkling the face of the Earth with seismographs, and hooking up these instruments to computers and satellites is not the answer either. ...1992, 14 April, Unpredicted earthquake of magnitude 6 in the heart of Europe, amids hundreds of seismographs, computers, and professors of seismology ... the strongest shock in central Europe since 1755".[25]

The time-energy accumulation law in the focus has been derived in 2006.[26] Then, the geometric parameter of the growth model has been introduced. The basic notion is the fundamental earthquakes, which imply an energy threshold and an accumulation time threshold. On this basis the Gutenberg-Richter distribution law in magnitude has been derived, together with the recurrence time. The basic parameters of background-earthquakes distribution have been indentified.[27] The

[22] M. Bath, "Lateral inhomogeneities of the upper mantle", Tectonophysics **2** 483 (1965); C. F. Richter, *Elementary Seismology*, Freeman, San Francisco, CA (1958) p. 69.

[23] L. Gulia and S. Wiemer, "Real-time discrimination of earthquake foreshocks and aftershocks", Nature **574** 193 (2019).

[24] F. Omori, "On the after-shocks of earthquakes", J. Coll. Sci. Imper. Univ. Tokyo **7** 111 (1894).

[25] A. Ben-Menahem, "A concise history of mainstream seismology: origins, legacy and perspectives", Bull. Sesim. Soc. Am. **85** 1202 (1995).

[26] B. F. Apostol, "Model of Seismic Focus and Related Statistical Distributions of Earthquakes", Phys. Lett. **A357** 462 (2006).

[27] B. F. Apostol, *Seismology*, Nova (2020); B. F. Apostol and L. C. Cune, "Entropy of earthquakes: application to Vrancea earthquakes", Acta Geophys. doi:

recurrence time, known also as the periodicity problem, is considered sometimes as a challenge to the modern Seismology. By updating the regular-earthquakes background parameters, we were able to identify an increase of the recurrence time of the big earthquakes in Vrancea. Further on, the Omori law was derived,[28] the correlated distribution was established and Bath's law, the roll-off effect and time-magnitude correlations were identified.[29] By making use of the time-magnitude correlations of the foreshocks a short-term prediction procedure has been established.[30]

15.5 Practical Seismology

We focus on a seismic region, where we have a large data set of earthquakes accumulated in a long period of time. The parameters of the magnitude (M) distribution of the regular (background) earthquakes (the Gutenberg-Richter distribution) are the slope β and the inverse of the seismicity rate t_0 ($-\ln t_0$). We update the Gutenberg-Richter distribution periodically (say, every five years), to see possible changes in these parameters. This helps to update the recurrence time ($t_r = t_0 e^{\beta M}$) of the big earthquakes.

The seismic activity is a non-equilibrium process, *i.e.* it is a process with a decreasing entropy ($S = 1 - \ln \beta$); therefore, the parameter β exhibits a slow increase in time, due to the accumulation of small-magnitude earthquakes, interrupted from time to time by big earthquakes, which decrease the parameter β. We update β weekly, for each year. If an increase appears in its steady slope, it follows that the non-equilibrium was accentuated, and we may expect big earthquakes.

The number of the successors of any (big, or moderate) earthquake (main shock) is distributed in time as an inverse-time law (Omori

10.1007/s11600-021-00550-4 (2021).

[28] B. F. Apostol, "Euler's transform and a generalized Omori's law", Phys. Lett. **A351** 175 (2006); B. F. Apostol and L. C. Cune, "Short-term seismic activity in Vrancea. Inter-event time distributions", Ann. Geophys. **63** SE328 (2020); doi: 10.4401/ag-8366.

[29] B. F. Apostol, "Correlations and Bath's law", Res. Geophys. **5** 100011 (2021).

[30] B. F. Apostol and L. C. Cune, "On the time variation of the Gutenberg-Richter parameter in foreshock sequences", J. Theor. Phys. **323** (2020).

law), for various magnitudes; this is a conditional probability distribution, for aftershocks. By updating periodically such distributions (say, every five years), we are able to tell what is the probability for an earthquake with a given magnitude to occur in the next one, two, three,... days after the occurrence of a main shock (next-earthquake distributions).

Some precursory seismic events (foreshocks) are correlated to the main shock. They show themselves as a short magnitude-descending sequence in the proximity of the main shock. Their magnitude obeys the law $M(t) = \frac{2}{3} \ln \frac{t_{ms}-t}{\tau_0}$, where t is the time till the occurrence moment t_{ms} of the main shock. The parameters t_{ms} and τ_0 are fitting parameters. By fitting this law to short magnitude-descending sequences of earthquakes, we can predict the occurrence time t_{ms} of the main shock. This is a short-time prediction, which may be attained by monitoring daily (hourly) the seismic activity. By using $\tau_0 = rt_0 e^{-b(1-r)M_0}$, where $r = \beta/b$, $b = 3.45$, we can find the magnitude M_0 of the main shock.

We have an approximate procedure of determining the earthquake parameters and the parameters of an earthquake focus. The mean seismic moment is given by $\overline{M} = 8\pi\rho c^2 (Rv)^{3/2}$, where ρ $(= 5.5g/cm^3)$ is the Earth's density, c $(= 5km/s)$ is a mean velocity of the seismic waves and v is a mean amplitude of the P and S seismic waves measured on Earth's surface at distance R from the focus. The duration T of the seismic activity in the focus is given approximately by $cT = (2Rv)^{1/2}$; the focal volume is $V = \pi(cT)^3$ $(l = V^{1/3}$ is the fault slip). The energy of the earthquake is $E = \overline{M}/2\sqrt{2}$, the moment magnitude M is given by $\log E = \frac{3}{2}M + 15.65$ and for a local magnitude we can use $M_l = M - 3$. By measuring v we can get an estimate of all these parameters. The procedure can also be applied to explosions (less the magnitude). A more exact procedure is given in the Earthquake Parameters chapter, for determining the tensor M_{ij} of the seismic moment $(\overline{M} = (M_{ij}^2)^{1/2})$, the fault slip and the fault orientation. The determination of earthquake parameters and the foreshock prediction can be reported periodically in a Seismological Bulletin.

The gradual accumulation of the seismic stress may be periodically discharged by small, static deformations of Earth's crust in the epicentral region. As shown in the Quasi-Static Deformations chapter, by measuring these static deformations we may give an estimate of an average seismic moment, the depth and the volume of the focus and even the tensor of the seismic moment. More important, by monitoring continuously the crustal deformations in a seismogenic zone, we can assess the possibility of the occurrence of an important earthquake, because after a silence period we may expect an important, sudden discharge.

The structures built on Earth's surface can be viewed as (embedded) elastic bars. Under the action of a seismic motion they develop vibrations, according to their eigenmodes. An interesting particularity is that these vibrations may exhibit an amplification factor $c/l\alpha$, where c is the velocity of the elastic wave, l is the length of the bar and α is the time attenuation factor. The data recorded by sensors installed in these buildings can be used to get information for the behaviour of such buildings under a seismic motion. A sub-surface local inhomogeneity acts as a buried bar. Under a seismic action, this inhomogeneity may vibrate, with a large amplification factor close to resonance. The measurements of such site amplification factors can give information about sub-surface inhomogeneities.

The spectral analysis of the local displacement (velocity, acceleration) produced by the P and S seismic waves exhibits a frequency maximum, which is related to the focal dimension and the (local) velocity of the elastic waves. This is the site (spectral) response; it depends on direction. The identification of this maximum gives information about the focal dimensions and local conditions. Site amplification factors may appear.

16 Appendix

16.1 Geometric-growth model of energy accumulation in focus

We consider a typical earthquake, with a small focal region localized in the solid crust of the Earth.[1] The dimension of the focal region is so small in comparison to our distance scale, that we may approximate the focal region by a point in an elastic body. The movement of the tectonic plates may lead to energy accumulation in this pointlike focus. The energy accumulation in the focus is governed by the continuity equation (energy conservation)

$$\frac{\partial E}{\partial t} = -\boldsymbol{v} gradE \ \ , \tag{16.1}$$

where E is the energy, t denotes the time and $\boldsymbol{v}$ is an accumulation velocity. For such a localized focus we may replace the derivatives in equation (16.1) by ratios of small, finite differences. For instance, we replace $\partial E/\partial x$ by $\Delta E/\Delta x$, for the coordinate x. Moreover, we assume that the energy is zero at the borders of the focus, such that $\Delta E = -E$, where E is the energy in the centre of the focus. Also, we assume a uniform variation of the coordinates of the borders of this small focal region, given by equations of the type $\Delta x = u_x t$, where $\boldsymbol{u}$ is a small displacement velocity of the medium in the focal region. The energy accumulated in the focus is gathered from the outer region of the focus, as expected. With these assumptions equation (16.1) becomes

$$\frac{\partial E}{\partial t} = \left(\frac{v_x}{u_x} + \frac{v_y}{u_y} + \frac{v_z}{u_z} \right) \frac{E}{t} \ . \tag{16.2}$$

[1] B. F. Apostol, "A model of seismic focus and related statistical distributions of earthquakes", Phys Lett **A357** 462 (2006).

Let us assume an isotropic motion without energy loss; then, the two velocities are equal, $\boldsymbol{v} = \boldsymbol{u}$, and the bracket in equation (16.2) acquires the value 3. In the opposite limit, we assume a one-dimensional motion. In this case the bracket in equation (16.2) is equal to unity. A similar analysis holds for a two dimensional accumulation process. In general, we may write equation (16.2) as

$$\frac{\partial E}{\partial t} = \frac{1}{r}\frac{E}{t} \ , \tag{16.3}$$

where r is an empirical (statistical) parameter; we expect it to vary approximately in the range (1/3, 1). We note that equation (16.3) is a non-linear relationship between t and E. The parameter r may give an insight into the geometry of the focal region. This is why we call this model a geometric-growth model of energy accumulation in the focal region.

The integration of equation (16.3) needs a cutoff (threshold) energy and a cutoff (threshold) time. During a short time t_0 a small energy E_0 is accumulated. In the next short interval of time this energy may be lost, by a relaxation of the focal region. Consequently, such processes are always present in a focal region, although they may not lead to an energy accumulation in the focus. We call them fundamental processes (or fundamental earthquakes, or E_0-seismic events). It follows that we must include them in the accumulation process, such that we measure the energy from E_0 and the time from t_0. The integration of equation (16.3) leads to the law of energy accumulation in the focus

$$t/t_0 = (E/E_0)^r \ . \tag{16.4}$$

The time t in this equation is the time needed for accumulating the energy E, which may be released in an earthquake (the accumulation time).

16.2 Gutenberg-Richter law. Time probability

The well-known Hanks-Kanamori law reads

$$\ln \overline{M} = const + bM \ , \tag{16.5}$$

where $\overline{M}$ is the seismic moment, M is the moment magnitude and $b = 3.45$ ($\frac{3}{2}$ for base 10). The relation $\overline{M} = 2\sqrt{2}E$ has been established, where $\overline{M} = \left(\sum_{ij} M_{ij}^2\right)^{1/2}$ (mean seismic moment), M_{ij} is the tensor of the seismic moment and E is the energy of the earthquake.[2] If we identify the mean seismic moment with $\overline{M}$ in equation (16.5) we can write

$$\ln E = const + bM \tag{16.6}$$

(another *const*), or

$$E/E_0 = e^{bM} \quad , \tag{16.7}$$

where E_0 is a threshold energy (related to *const*). Making use of equation (16.4), we get

$$t = t_0 e^{brM} = t_0 e^{\beta M} \quad , \tag{16.8}$$

where $\beta = br$. From this equation we derive the useful relations $dt = \beta t_0 e^{\beta M} dM$, or $dt = \beta t dM$. If we assume that the earhquakes are distributed according to the well-known Gutenberg-Richter distribution,

$$dP = \beta e^{-\beta M} dM \quad , \tag{16.9}$$

we get the time distribution

$$dP = \beta \frac{t_0}{t} \frac{1}{\beta t} dt = \frac{t_0}{t^2} dt \ . \tag{16.10}$$

This law shows that the probability for an earthquake to occur between t and $t + dt$ is $\frac{t_0}{t^2} dt$; since the accumulation time is t, the earthquake has an energy E and a magnitude M given by the above formulae (equations (16.7) and (16.8)). The law given by equation (16.10) is also derived from the definition of the probability of the fundamental E_0-seismic events ($dP = -\frac{\partial}{\partial t}\frac{t_0}{t} dt$).[3] We note that this probability assumes independent earthquakes.

In addition, the definition of the entropy $S = -\int dM \rho \ln \rho$ with the distribution $\rho = dP/dM$ given by equation (16.9) leads to the entropy

[2] B. F. Apostol, "An inverse problem in seismology: derivation of the seismic source parameters from P and S seismic waves", J. Seismol. **23** 1017 (2019).

[3] B. F. Apostol, "Correlations and Bath's law", Results in Geophysical Sciences **5** 100011 (2021).

$S = 1 - \ln \beta$.[4] A series expansion in powers of a small subsequent time t of the time distribution given by equation (16.10) (conditional probability) leads to Omori's law.[5]

16.3 Correlations. Time-magnitude correlations

If two earthquakes are mutually affected by various conditions, and such an influence is reflected in the above equations, we say that they are correlated to each other. Also, we say that either one earthquake is correlated to the other. Of course, multiple correlations may exist, *i.e.* correlations between three, four, etc earthquakes. We limit ourselves to two-earthquake (pair) correlations. Very likely, correlated earthquakes occur in the same seismic region and in relatively short intervals of time. The physical causes of mutual influence of two earthquakes are various. Three types of earthquake correlations are identified.[6] In one type the neighbouring focal regions may share (exchange, transfer) energy. Since the energy accumulation law is non-linear, this energy sharing affects the occurrence time. We call these correlations time-magnitude correlations. They are a particular type of dynamical correlations. In a second type of correlations two earthquakes may share their accumulation time, which affects their total energy. We call such correlations (purely) dynamical correlations. Both these correlations affect the earthquake statistical distributions; in this respect, they are also statistical correlations. Finally, additional constraints on the statistical variables (*e.g.*, the magnitude of the accompanying seismic event be smaller than the magnitude of the main shock) give rise to purely statistical correlations.

Let an amount of energy E, which may be accumulated in time t,

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[4] B. F. Apostol and L. C. Cune, "Entropy of earthquakes: application to Vrancea earthquakes", Acta Geophys. doi: 10.1007/s11600-021-00550-4 (2021).

[5] B. F. Apostol, "Euler's transform and a generalized Omori's law", Phys. Lett. **A351** 175 (2006); B. F. Apostol and L. C. Cune, "Short-term seismic activity in Vrancea. Inter-event time distributions", Ann. Geophys. **63** SE328 (2020); doi: 10.4401/ag-8366.

[6] B. F. Apostol, "Correlations and Bath's law", Results in Geophysical Sciences **5** 100011 (2021).

be released by two successive earthquakes with energies $E_{1,2}$, such as $E = E_1 + E_2$ (energy sharing). According to the accumulation law (equation (16.4))

$$t/t_0 = (E/E_0)^r = (E_1/E_0 + E_2/E_0)^r = \\ = (E_1/E_0)^r (1 + E_2/E_1)^r \ , \tag{16.11}$$

or

$$t = t_1 \left[1 + e^{b(M_2 - M_1)}\right]^r \ , \tag{16.12}$$

where $t_1 = t_0(E_1/E_0)^r$ is the accumulation time of the earthquake with energy E_1 and magnitude M_1, and M_2 is the magnitude of the earthquake with energy E_2. From equation (16.12) we get

$$b(M_2 - M_1) = \ln\left[(1 + \tau/t_1)^{1/r} - 1\right] \ , \tag{16.13}$$

where $t = t_1 + \tau$, τ being the time elapsed from the occurrence of the earthquake 1 until the occurrence of the earthquake 2. If $\tau/t_1 \ll 1$, as in foreshock-main shock-aftershock sequences, this equation gives, after some simple manipulations,

$$M_2 = \frac{1}{b}\ln\frac{\tau}{\tau_0} \ , \ \tau_0 == rt_0 e^{-b(1-r)M_1} \ . \tag{16.14}$$

We can see that τ differs from the accumulation time of the earthquake 2 (compare to equation (16.8)); it is given by parameters which depend on the earthquake 1 (M_1). If the earthquake 1 is viewed as a main shock, then the earthquake 2 is a foreshock or an aftershock. These accompanying earthquakes are correlated to the main shock (and the main shock is correlated to them).

16.4 Correlations. Dynamical correlations

Let us assume that an earthquake occurs in time t_1 and another earthquake follows in time t_2. The total time is $t = t_1 + t_2$, so these earthquakes share their accumulation time, which affects their total energy. These are (purely) dynamical correlations. According to equation

(16.10) (and the definition of the probability), the probability density of such an event is given by

$$-\frac{\partial}{\partial t_2}\frac{t_0}{(t_1+t_2)^2}=\frac{2t_0}{(t_1+t_2)^3} \tag{16.15}$$

(where $t_0 < t_1 < +\infty$, $0 < t_2 < +\infty$). By passsing to magnitude distributions ($t_{1,2}=t_0e^{\beta M_{1,2}}$), we get

$$d^2P=4\beta^2\frac{e^{\beta(M_1+M_2)}}{\left(e^{\beta M_1}+e^{\beta M_2}\right)^3}dM_1dM_2 \tag{16.16}$$

(where $0 < M_{1,2} < +\infty$, corresponding to $t_0 < t_{1,2} < +\infty$, which introduces a factor 2 in equation (16.15)). This formula is a pair, bivariate statistical distribution. By using the pair correlations the Bath's law was established and the foreshock short-term prediction procedure was devised.[7] If we integrate this equation with respect to M_2, we get the distribution of a correlated earthquake (marginal distribution)

$$dP=\beta e^{-\beta M_1}\frac{2}{\left(1+e^{-\beta M_1}\right)^2}dM_1 \ ; \tag{16.17}$$

if we integrate further this distribution from $M_1 = M$ to $+\infty$, we get the correlated cumulative distribution

$$P(M)=\int_M^\infty dP=e^{-\beta M}\frac{2}{1+e^{-\beta M}} \ . \tag{16.18}$$

For $M \gg 1$ the correlated distribution becomes $P(M) \simeq 2e^{-\beta M}$ and $\ln P(M) \simeq \ln 2 - \beta M$, which shows that the slope β of the logarithm of the independent cumulative distribution (Gutenberg-Richter, standard distribution $e^{-\beta M}$) is not changed (for large magnitudes); the correlated distribution is only shifted upwards by $\ln 2$. On the contrary, for small magnitudes ($M \ll 1$) the slope of the correlated distribution becomes $\beta/2$ ($P(M) \simeq 1 - \frac{1}{2}\beta M + ...$ by a series expansion of equation (16.18)), instead of the slope β of the Gutenberg-Richter distribution ($e^{-\beta M} \simeq 1 - \beta M + ...$). The correlations modify the slope of the Gutenberg-Richter standard distribution for small magnitudes. This is the roll-off effect.

[7]B. F. Apostol, *loc. cit.*

16.5 Seismic waves and main shock

The components of the seismic tensorial point force (per unit volume) are

$$F_i(\mathbf{R}, t) = M_{ij} T \delta(t) \partial_j \delta(\mathbf{R} - \mathbf{R}_0) \ , \tag{16.19}$$

where M_{ij} is the tensor of the seismic moment of the focus placed at $\boldsymbol{R}_0$, for a time-impulse seismic activity which lasts a short time T. The far-field seismic waves produced by this force are[8]

$$\begin{aligned} u_i^f(\boldsymbol{R}, t) = \tfrac{T}{4\pi c_t} \tfrac{m_{ij} x_j}{R^2} \delta'(R - c_t t) + \\ + \tfrac{T}{4\pi} \tfrac{m_{jk} x_i x_j x_k}{R^4} \left[\tfrac{1}{c_l} \delta'(R - c_l t) - \tfrac{1}{c_t} \delta'(R - c_t t) \right] \ , \end{aligned} \tag{16.20}$$

where $m_{ij} = M_{ij}/\rho$, ρ is the density of the body, x_i are the cartesian coordinates of the position $\boldsymbol{R}$ and $c_{l,t}$ are the velocities of the elastic waves. In this equation we can identify immediately the P (longitudinal) and S (transverse) seismic waves (primary waves). The derivation of this solution of the Navier-Cauchy equation requires the regulariztion of the Helmhotz, or Hertz, potentials.[9]

The displacement in the main shock on Earth's surface is given by

$$\begin{aligned} u_r \simeq \tfrac{\chi_0 \tau_l}{4 c_l} \cdot \tfrac{r}{\left(c_l^2 \tau_l^2 - r^2\right)^{3/2}} \ , \\ u_\varphi \simeq -\tfrac{h_{0z} \tau_t}{4 c_t} \cdot \tfrac{r}{(c_t^2 \tau_t^2 - r^2)^{3/2}} \ , \\ u_z \simeq \tfrac{h_{0\varphi} \tau_t}{4 c_t} \cdot \tfrac{c_t^2 \tau_t^2}{r (c_t^2 \tau_t^2 - r^2)^{3/2}} \end{aligned} \tag{16.21}$$

in cylindrical coordinates, where r is the epicentral distance, $\tau_{l,t}$ are times slightly smaller than the time from the moment the primary waves reach the epicentre and χ_0, $\boldsymbol{h}_0$ are Helmholtz potentials of the order of $M/4\pi\rho R l^2$, where M is the magnitude of the seismic moment

[8] B. F. Apostol, "Elastic waves inside and on the surface of a half-space", Quart. J. Mech. Appl. Math. **70** 289 (2017).

[9] M. Apostol, "On unphysical terms in the elastic Hertz potentials", Acta Mech. **228** 2733 (2017).

and l is the dimension of the focus. We can see the abrupt wall and the long tail in this main shock; actually, there are two main shocks, slightly displaced, corresponding to the difference between $c_l\tau_l$ and $c_t\tau_t$. Equations (16.21) are valid for intermediate distances of the order of the depth z_0 of the focus.[10]

The near-fied displacement is given by

$$u_i^n = -\frac{T}{4\tau c_t}\frac{m_{ij}x_j}{R^3}\delta(R-c_tt)+$$
$$+\frac{T}{8\pi R^3}\left(m_{jj}x_i + 4m_{ij}x_j - \frac{9m_{jk}x_ix_jx_k}{R^2}\right)\cdot \tag{16.22}$$
$$\cdot\left[\frac{1}{c_l}\delta(R-c_lt)-\frac{1}{c_t}\delta(R-c_tt)\right] \ .$$

The deformations produced in homogeneous and isotropic half-space by a static tensorial force given by equation (16.19) have also been computed.[11]

16.6 Earthquake parameters

Apart from a consistently covariant procedure for detrmining exactly the seismic moment and the earthquake parameters, we have also an approximate procedure. The mean seismic moment is given by $\overline{M} = 8\pi\rho c^2(Rv)^{3/2}$, where ρ $(= 5.5g/cm^3)$ is the Earth's density, c $(= 5km/s)$ is a mean velocity of the seismic waves and v is a mean amplitude of the P and S seismic waves measured on Earth's surface at distance R from the focus. The duration T of the seismic activity in the focus is given approximately by $cT = (2Rv)^{1/2}$; the focal volume is $V = \pi(cT)^3$ ($l = V^{1/3}$ is the fault slip). The energy of the earthquake is $E = \overline{M}/2\sqrt{2}$, the moment magnitude M is given by $\log E = \frac{3}{2}M + 15.65$ and for a local magnitude we can use $M_l = M-3$. By measuring v we can get an estimate of all these parameters. The procedure can also be applied to explosions (less the magnitude).[12]

[10] B. F. Apostol, *loc. cit.*

[11] B. F. Apostol, "Elastic displacement in a half-space under the action of a tensor force. General solution for the half-space with point forces", J. Elast. **126** 231 (2017).

[12] B. F. Apostol, *Seismology*, Nova (2020).

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